Journal of Abstract and Computational Mathematics
http://www.ntmsci.com/jacm

Fixed point theorems for some multi-valued contraction mappings defined in partial Hausdorff metric spaces

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Received: 16 May 2019, Accepted: 24 July 2020
Published online: 15 October 2020.

Abstract: In this paper, we give some fixed point theorems for satisfying different contractive conditions on complete partial Hausdorff metric spaces. Also, we prove some fixed point theorems for two operators that do not necessarily commute with each other to have a common fixed point as in metric spaces. We also state an example in support of our conclusions.

Keywords: Partial metric space, partial Hausdorff metric, fixed point, multi-valued mappings.

1 Introduction

The notion of a partial metric space has been firstly introduced by Matthews, [9]. In [9], he extended Banach contraction principle in the setting of complete partial metric space. Then, further fixed point theorems of partial metric space given by many authors ([2],[6],[7] and [4]). In [3], they proved the some generalized versions of the fixed point theorem of Matthews and established a homotopy results. Based on the partial metric on a set X, in [1], they presented a notion of partial Hausdorff metric on the $\Omega_C(X)$. Besides, in [1], they studied of fixed point theorem for multi-valued mappings on a partial metric space using the partial Hausdorff metric and generalized Nadler's fixed point theorem.

Let $\Omega_C^p(X)$ be the family of all nonempty closed and bounded subsets of a partial metric space (X,p). The purpose of this paper is to investigate two mappings $T: X \to \Omega_C^p(X)$ (for a partial metric space X) which satisfy the following contractive definitions:

$$H_p(Tx,Ty) \leq ap(x,y) + b[p(x,Tx) + p(y,Ty)] + c[p(x,Ty) + p(y,Tx)]$$

and

$$\begin{aligned} H_p(Tx,Ty) \leq & a \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\right\} \\ & + b\max\{p(x,Tx), p(y,Ty)\} + c[p(x,Ty) + p(y,Tx)] \end{aligned}$$

where for all $x,y \in X$, where a,b,c are nonnegative real numbers such that $a+2b+2c<1$ and $a+b+2c<1$, respectively. Also, another purpose of this study is to prove common fixed point theorems for two set-valued mappings defined in a partial Hausdorff metric space.

Firstly, we recall some definitions and some related results from partial metric space.

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Definition 1.[9] *Let X be a nonempty set. A mapping $p : X \times X \to \mathbb{R}^+$ is a partial metric on X, if for all $x,y,z \in X$. We have*

(p1) $p(x,x) = p(y,y) = p(x,y)$ *if and only if* $x = y$,
(p2) $p(x,x) \leq p(x,y)$,
(p3) $p(x,y) = p(y,x)$,
(p4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is then callad a partial metric space.

If $p(x,y) = 0$, then p1) and p2) imply that $x = y$. But the reverse does not satisfy always.

Example 1.[9] Let $X = \{[a,b] : a,b \in \mathbb{R}, a \leq b\}$, that is, $X = \Omega_C(\mathbb{R})$ and define a function $p : X \times X \to \mathbb{R}^+$ is defined as $p(x,y) = p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$, then (X,p) is a partial metric space. [9] Every metric space is a partial metric space. Each partial metric p on X generates a T_0 topology τ_p on X with as a base the family of the open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$.

Definition 2.[9] *Let (X,p) be a partial metric space. Then:*

(a) *A sequence (x_n) in (X,p) converges to a point $x \in X$ with respect to τ_p if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.*
(b) *A sequence (x_n) in (X,p) is called Cauchy sequence if there exists and is finite $\lim_{n,m\to\infty} p(x_n,x_m)$.*
(c) *A partial metric space (X,p) is called a complete partial metric space if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$.*

Remark.[9]Let (X,p) be a partial metric space. Then the function $d_p : X \times X \to [0,\infty)$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Let (X,p) be a partial metric space, a sequence $\{x_n\}$ in (X,d_p) is said to be convergent to a point $x \in X$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m). \tag{1}$$

Lemma 1.[1] *Let (X,p) be a partial metric space. Then:*

(a) *A sequence (x_n) in X is a Cauchy with respect to p if and only if it is Cauchy with respect to d_p.*
(b) *A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete.*

Now, let us give the definition of partial Hausdorff metric space and reletad results. First, we remember and state the definition of Hausdorff metric for metric spaces.

Let (X,d) be a metric space and $\Omega_C(X)$ denotes the collection of all nonempty closed and bounded subsets of X. For $A,B \in \Omega_C(X)$, define

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(x,A) = \inf\{d(x,a) : a \in A\}$ is the distance of a point x to the set A. We know that H is a metric on $\Omega_C(X)$, called the Hausdorff metric induced by the metric d.

[1] In a partial metric space closedness is taken from (X,τ_p) and boundedness is given as follows: A is a bounded subset in (X,p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0,a) < p(a,a) + M$. Again in [1] following nations are defined. For $A,B \in \Omega_C^p(X)$ and $x \in X$,

$$p(x,A) = \inf\{p(x,a), a \in A\} \text{ and } \delta_p(A,B) = \sup\{p(a,B) : a \in A\}.$$

From here for the functions $\delta_p : \Omega_C^p(X) \times \Omega_C^p(X) \to \mathbb{R}^+$ and $H_p : \Omega_C^p(X) \times \Omega_C^p(X) \to \mathbb{R}^+$, we have the following

$$H_p(A,B) = \max\left\{\delta_p(A,B), \delta_p(B,A)\right\}.$$

Remark.[**3**] Let (X,p) be a partial metric space and A be any nonempty subset of X, then

$$a \in \overline{A} \text{ if and only if } p(a,A) = p(a,a). \tag{2}$$

Proposition 1.[*1*] *Let $\delta_p : \Omega_C^p(X) \times \Omega_C^p(X) \to \mathbb{R}^+$. For all $A,B,C \in \Omega_C^p(X)$, we have the following:*

(i) $\delta_p(A,A) = \sup\{p(a,a) : a \in A\}$,
(ii) $\delta_p(A,A) \leq \delta_p(A,B)$,
(iii) $\delta_p(A,B) = 0$ *implies that* $A \subseteq B$,
(iv) $\delta_p(A,B) \leq \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c)$.

Proposition 2.[*1*] *Let (X,p) be a partial metric space. For any $A,B,C \in \Omega_C^p(X)$, we have*

(1) $H_p(A,A) \leq H_p(A,B)$,
(2) $H_p(A,B) = H_p(A,B)$,
(3) $H_p(A,B) \leq H_p(A,C) + H_p(C,B) - \inf_{c\in C} p(c,c)$.

We know that a multi-valued mapping $T : X \to \Omega_C(X)$ is said to be contraction if

$$H(Tx,Ty) \leq kd(x,y)$$

for all $x,y \in X$ and for some $k \in [0,1)$. After above definition Nadler (in [8]) was proved the following theorem.

Theorem 1.[*8*] *Let (X,d) be a complete metric space and $T : X \to \Omega_C(X)$ be a contraction mapping. Then, there exists $x \in X$ such that $x \in Tx$.*

Lemma 2.[***1***] *Let (X,p) be a partial metric space, $A,B \in \Omega_C^p(X)$ and $h > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that*

$$p(a,b) \leq hH_p(A,B).$$

Theorem 2.[*1*] *Let (X,p) be a complete partial metric space. If $T : X \to \Omega_C(X)$ is a multi-valued mapping such that for all $x,y \in X$, we have*

$$H_p(Tx,Ty) \leq kp(x,y)$$

where $k \in (0,1)$. Then T has a fixed point.

2 Main Results

In this section, we give some fixed point theorems for multi-valued mappings on a complete partial metric space.

Theorem 3. *Let (X,p) be a complete partial metric space and let $T : X \to \Omega_C^p(X)$ be a map such that*

$$H_p(Tx,Ty) \le ap(x,y) + b[p(x,Tx) + p(y,Ty)] \\ +c[p(x,Ty) + p(y,Tx)] \tag{3}$$

for all $x, y \in X$ where $a,b,c \ge 0$ and $a+2b+2c < 1$. Then T has a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. Since $k = a+2b+2c < 1$, then $\frac{1}{\sqrt{k}} > 1$. Thus, by using Lemma 2, we have $x_2 \in Tx_1$ such that

$$p(x_1,x_2) \le \frac{1}{\sqrt{k}} H_p(Tx_0,Tx_1). \tag{4}$$

From (3) and (4), we get

$$\begin{aligned} p(x_1,x_2) &\le \frac{1}{\sqrt{k}} \begin{bmatrix} ap(x_0,x_1) + b[p(x_0,x_1) + p(x_1,x_2)] \\ +c[p(x_0,x_2) + p(x_1,x_1)] \end{bmatrix} \\ &\le \frac{1}{\sqrt{k}} [(a+b+c)p(x_0,x_1) + (b+c)p(x_1,x_2)] \\ &\le \frac{\sqrt{k}}{1-\sqrt{k}} p(x_0,x_1) \\ &\le \sqrt{k} p(x_0,x_1). \end{aligned} \tag{5}$$

From (5), we have

$$p(x_1,x_2) \le \sqrt{k} p(x_0,x_1) < p(x_0,x_1). \tag{6}$$

If $x_1 = x_2$, then x_1 is a fixed point. Suppose that $x_1 \ne x_2$. Again, from (3), we get

$$H_p(Tx_1,Tx_2) \le ap(x_1,x_2) + b[p(x_1,Tx_1) + p(x_2,Tx_2)] \\ +c[p(x_1,Tx_2) + p(x_2,Tx_1)]. \tag{7}$$

Again by using Lemma 2, we have $x_3 \in Tx_2$ such that

$$p(x_2,x_3) \le \frac{1}{\sqrt{k}} H_p(Tx_1,Tx_2). \tag{8}$$

From (7) and (8), we get

$$\begin{aligned} p(x_2,x_3) &\le \frac{1}{\sqrt{k}} \begin{bmatrix} ap(x_1,x_2) + b[p(x_1,x_2) + p(x_2,x_3)] \\ +c[p(x_1,x_3) + p(x_2,x_2)] \end{bmatrix} \\ &\le \frac{1}{\sqrt{k}} [(a+b+c)p(x_1,x_2) + (b+c)p(x_2,x_3)] \\ &\le \frac{\sqrt{k}}{1-\sqrt{k}} p(x_1,x_2) \\ &\le \sqrt{k} p(x_1,x_2). \end{aligned} \tag{9}$$

Now, from (9) and mathematical induction, we obtain

$$p(x_2,x_3) \le \sqrt{k} p(x_1,x_2) \le \sqrt{k}\sqrt{k} p(x_0,x_1). \tag{10}$$

Continuing the same way, we get $\{x_n\} \subset X$ such that $x_{n-1} \in Tx_n$ and $x_{n-1} \neq x_n$, with

$$p(x_n, x_{n+1}) \leq \left(\sqrt{k}\right)^n p(x_0, x_1) \text{ for all } n \in \mathbb{N}. \tag{11}$$

Using (11) and the property (p4) of a partial metric, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} p(x_n, x_{n+m}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{n+m-1}, x_{n+m}) \\ &\leq \left(\sqrt{k}\right)^n p(x_0, x_1) + \left(\sqrt{k}\right)^{n+1} p(x_0, x_1) + \ldots \\ &\quad + \left(\sqrt{k}\right)^{n+m-1} p(x_0, x_1) \\ &= \left(\left(\sqrt{k}\right)^n + \left(\sqrt{k}\right)^{n+1} + \ldots + \left(\sqrt{k}\right)^{n+m-1}\right) p(x_0, x_1) \\ &\leq \frac{\left(\sqrt{k}\right)^n}{1-\sqrt{k}} p(x_0, x_1) \to 0 \text{ as } n \to +\infty \end{aligned} \tag{12}$$

since $0 < k < 1$. This shows that $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is a complete partial metric space, by Lemma 1, (X, p^s) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to some $x \in X$ with respect to the metric p^s, that is, $\lim_{n\to\infty} p^s(x_n, x) = 0$. Again from (1), we have

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = 0. \tag{13}$$

Now, we show that $p(x, Tx) = 0$. On the contrary, suppose that $p(x, Tx) > 0$. By using the (p4) inequality and (3), we have

$$\begin{aligned} p(x, Tx) &\leq p(x, x_{n+1}) + p(x_{n+1}, Tx) - p(x_{n+1}, x_{n+1}) \\ &\leq p(x, x_{n+1}) + p(Tx_n, Tx) \\ &\leq p(x, x_{n+1}) + ap(x_n, x) + b[p(x_n, Tx_n) + p(x, Tx)] \\ &\quad + c[p(x_n, Tx) + p(x, Tx_n)] \end{aligned}$$

letting $n \to \infty$, we get

$$p(x, Tx) \leq (b + c)\, p(x, Tx).$$

But this is impossible for $a, b, c \geq 0$ and $a + 2a + 2b < 1$. Thus, $p(x, Tx) = 0$. Therefore, from (13), we get

$$p(x, Tx) = 0 = p(x, x),$$

which from (2) implies that $x \in Tx$.

Now, we give the illustrative example.

Let $X = \{0, 1, 2\}$ be endowed with the partial metric $p : X \times X \to \mathbb{R}^+$ defined by $p(x, y) = \frac{1}{4}\max\{x, y\}$ for all $x, y \in X$. Clearly, p is a not metric on X because of $p(1, 1) = \frac{1}{4} \neq 0$ and $p(2, 2) = \frac{1}{2} \neq 0$. Further, (X, p) complete partial metric space. Define the mapping $T : X \to \Omega_C^p(X)$ by $T(0) = T(1) = \{0\}$ and $T(2) = \{0, 1\}$. Tx is closed and bounded for all $x \in X$ under the given partial metric space (X, p). Now, we shall show that, for all $x, y \in X$, the contractive condition (2.1) is satisfied with $a = \frac{1}{3}, b = \frac{1}{6}$ and $c = \frac{1}{8}$. We consider the following three cases:

(i) $x, y \in \{0, 1\}$. We obtain $H_p(T(x), T(y)) = H_p(\{0\}, \{0\}) = 0$. Thus (2.1) is satisfied.

(ii) $x \in \{0,1\}$, $y = 2$. We obtain

$$\begin{aligned} H_p(T(0), T(2)) &= H_p(T(1), T(2)) \\ &= H_p(\{0\}, \{0,1\}) \\ &= \frac{1}{4} \\ &\leq ap(0,2) + b[p(0,T(0)) + p(2,T(2))] \\ &\quad + c[p(0,T(2)) + p(2,T(0))] \\ &= \frac{a}{2} + \frac{b}{2} + \frac{3c}{4} \\ &\leq ap(1,2) + b[p(1,T(1)) + p(2,T(2))] \\ &\quad + c[p(1,T(2)) + p(2,T(1))] \\ &= \frac{a}{2} + \frac{3b}{2} + \frac{3c}{4}. \end{aligned}$$

(iii) $x = y = 2$. We get

$$\begin{aligned} H_p(T(2), T(2)) &= H_p(\{0,1\}, \{0,1\}) \\ &= \frac{1}{4} \\ &\leq ap(2,2) + b[p(2,T(2)) + p(2,T(2))] \\ &\quad + c[p(2,T(2)) + p(2,T(2))] \\ &= \frac{a}{2} + b + c. \end{aligned}$$

Thus, all conditions of Theorem 7 are satisfied. Here, $x = 0$ is a fixed point of T.

Now we state another fixed point theorem on a complete partial metric space.

Theorem 4. *Let (X,p) be a complete partial metric space and let $T : X \to \Omega_C^p(X)$ be a map such that*

$$\begin{aligned} H_p(Tx, Ty) \leq\ & a \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty) + p(y,Tx)}{2}\right\} \\ & + b \max\{p(x,Tx), p(y,Ty)\} + c[p(x,Ty) + p(y,Tx)] \end{aligned} \tag{14}$$

for all $x, y \in X$ where a, b, c are nonnegative real numbers such that $a + b + 2c < 1$. Then T has a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. From Lemma 2 with $k = a + b + 2c$ and $h = \frac{1}{\sqrt{k}}$ there exists $x_2 \in Tx_1$ such that

$$p(x_1, x_2) \leq \frac{1}{\sqrt{k}} H_p(Tx_0, Tx_1). \tag{2.13}$$

From (14) and (2.13), we get

$$\begin{aligned} p(x_1,x_2) &\leq \frac{1}{\sqrt{k}} \begin{bmatrix} a \max\left\{p(x_0,x_1), p(x_1,x_2), \frac{p(x_0,x_2)+p(x_1,x_1)}{2}\right\} \\ +b \max\{p(x_0,x_1), p(x_1,x_2)\} \\ +c[p(x_0,x_2) + p(x_1,x_1)] \end{bmatrix} \\ &\leq \frac{1}{\sqrt{k}} \begin{bmatrix} a \max\{p(x_0,x_1), p(x_1,x_2)\} \\ +b \max\{p(x_0,x_1), p(x_1,x_2)\} \\ c[p(x_0,x_1) + p(x_1,x_2)] \end{bmatrix} \\ &\leq \frac{1}{\sqrt{k}} (a + b + 2c) \max\{p(x_0,x_1), p(x_1,x_2)\} \\ &\leq \sqrt{k} \max\{p(x_0,x_1), p(x_1,x_2)\}. \end{aligned} \tag{15}$$

If we assume that $\max\{p(x_0,x_1),p(x_1,x_2)\} = p(x_1,x_2)$, then we get a contradiction to (15). Thus, $\max\{p(x_0,x_1),p(x_1,x_2)\} = p(x_0,x_1)$. From (15), we have

$$p(x_1,x_2) \leq \sqrt{k}p(x_0,x_1) < p(x_0,x_1). \tag{16}$$

If $x_1 = x_2$, then x_1 is a fixed point. Suppose that $x_1 \neq x_2$. Again, from (14), we get

$$\begin{aligned} H_p(Tx_1,Tx_2) \leq\ & a\max\left\{\begin{array}{c} p(x_1,x_2), p(x_1,Tx_1), p(x_2,Tx_2), \\ \frac{p(x_1,Tx_2)+p(x_2,Tx_1)}{2} \end{array}\right\} \\ & +b\max\{p(x_1,Tx_1),p(x_2,Tx_2)\} \\ & +c[p(x_1,Tx_2)+p(x_2,Tx_1)] \end{aligned} \tag{17}$$

Again by using Lemma 2, we have $x_3 \in Tx_2$ such that

$$p(x_2,x_3) \leq \frac{1}{\sqrt{k}}H_p(Tx_1,Tx_2). \tag{18}$$

From (17) and (18), we get

$$\begin{aligned} p(x_2,x_3) &\leq \frac{1}{\sqrt{k}}\left[\begin{array}{c} a\max\left\{p(x_1,x_2),p(x_2,x_3),\frac{p(x_1,x_3)+p(x_2,x_2)}{2}\right\} \\ +b\max\{p(x_1,x_2),p(x_2,x_3)\} \\ +c[p(x_1,x_3)+p(x_2,x_2)] \end{array}\right] \\ &\leq \frac{1}{\sqrt{k}}\left[\begin{array}{c} a\max\{p(x_1,x_2),p(x_2,x_3)\} \\ +b\max\{p(x_1,x_2),p(x_2,x_3)\} \\ c[p(x_1,x_2)+p(x_2,x_3)] \end{array}\right] \\ &\leq \frac{1}{\sqrt{k}}(a+b+2c)\max\{p(x_1,x_2),p(x_2,x_3)\} \\ &\leq \sqrt{k}\max\{p(x_1,x_2),p(x_2,x_3)\}. \end{aligned} \tag{19}$$

If we assume that $\max\{p(x_1,x_2),p(x_2,x_3)\} = p(x_2,x_3)$, then we get a contradiction to (19). Thus, $\max\{p(x_1,x_2),p(x_2,x_3)\} = p(x_1,x_2)$. From (19) and (16) , we have

$$p(x_2,x_3) \leq \sqrt{k}p(x_1,x_2) < \sqrt{k}\sqrt{k}p(x_0,x_1). \tag{20}$$

Continue similar to the proof of the above theorem we get $\{x_n\} \subset X$ such that $x_{n-1} \in Tx_n$ and $x_{n-1} \neq x_n$, with

$$p(x_n,x_{n+1}) \leq \left(\sqrt{k}\right)^n p(x_0,x_1) \text{ for all } n \in \mathbb{N}. \tag{21}$$

Using (21) and the property (p4) of a partial metric, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} p(x_n,x_{n+m}) &\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \ldots + p(x_{n+m-1},x_{n+m}) \\ &\leq \left(\sqrt{k}\right)^n p(x_0,x_1) + \left(\sqrt{k}\right)^{n+1} p(x_0,x_1) + \ldots \\ &\quad + \left(\sqrt{k}\right)^{n+m-1} p(x_0,x_1) \\ &= \left(\left(\sqrt{k}\right)^n + \left(\sqrt{k}\right)^{n+1} + \ldots + \left(\sqrt{k}\right)^{n+m-1}\right) p(x_0,x_1) \\ &\leq \frac{\left(\sqrt{k}\right)^n}{1-\sqrt{k}} p(x_0,x_1) \to 0 \text{ as } n \to +\infty \end{aligned} \tag{22}$$

since $0 < k < 1$. This shows that $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is a complete partial metric space, by Lemma 1, (X, p^s) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to some $x \in X$ with respect to the metric p^s, that is, $\lim_{n\to\infty} p^s(x_n, x) = 0$. Again from (1), we have

$$p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = 0. \tag{23}$$

Now, we show that $p(x, Tx) = 0$. On the contrary, suppose that $p(x, Tx) > 0$. By using the triangular inequality and (14), we have

$$\begin{aligned} p(x,Tx) &\le p(x,x_{n+1}) + p(x_{n+1},Tx) - p(x_{n+1},x_{n+1}) \\ &\le p(x,x_{n+1}) + p(Tx_n,Tx) \\ &\le p(x,x_{n+1}) + a\max\left\{ \begin{array}{c} p(x_n,x), p(x_n,Tx_n), p(x,Tx), \\ \frac{p(x_n,Tx)+p(x,Tx_n)}{2} \end{array} \right\} \\ &\quad + b\max\{p(x_n,Tx_n), p(x,Tx)\} + c[p(x_n,Tx) + p(x,Tx_n)] \end{aligned}$$

letting $n \to \infty$, we get

$$p(x,Tx) \le (a+b+c)\, p(x,Tx).$$

But this is impossible for $a,b,c \ge 0$ and $a + 2a + 2b < 1$. Thus, $p(x,Tx) = 0$. Therefore, from (23), we get

$$p(x,Tx) = 0 = p(x,x),$$

which from (2) implies that $x \in Tx$.

Now, we remember that the definition of a common fixed point theorem of two set-valued mappings.

Definition 3. *Let (X, p) be a partial metric space and $S, T : X \to \Omega_C^p(X)$. A point $x \in X$ is said to be a common fixed point of S and T, that is $x \in Tx$ and $x \in Sx$.*

Theorem 5. *Suppose that (X, p) be a partial metric space and $S, T : X \to \Omega_C^p(X)$. If there exists $a + 2b + 2c = r \in [0,1)$ such that*

$$H_p(Tx, Sy) \le ap(x,y) + b[p(x,Tx) + p(y,Sy)] + c[p(x,Sy) + p(y,Tx)] \tag{24}$$

for any $x, y \in X$. Then there exist $z \in X$ such that $z \in Tz$ and $z \in Sz$.

Proof. Let x_0 be an arbitrary point in X. We can find x_1 and x_2 in X such that $x_2 \in Tx_1$ and $x_1 \in Sx_0$. In general, $x_{2n} \in X$ is chosen such that

$$x_{2n+2} \in Tx_{2n+1} \text{ and } x_{2n+1} \in Sx_{2n} \text{ for } n = 0, 1, 2, \ldots$$

If there exists a positive m such that $x_{2m} = x_{2m+1}$, then $x_{2m} = x_{2m+1} \in Sx_{2m}$. From (24), we get

$$\begin{aligned} H_p(Tx_{2m+1}, Sx_{2m}) &\le ap(x_{2m+1}, x_{2m}) + b[p(x_{2m+1}, Tx_{2m+1}) + p(x_{2m}, Sx_{2m})] \\ &\quad + c[p(x_{2m+1}, Sx_{2m}) + p(x_{2m}, Tx_{2m+1})]. \end{aligned}$$

Then there exists $r<1$ such that $\frac{1}{\sqrt{r}}>1$. Thus by Lemma 1, we have $x_{2m+2}\in Tx_{2m+1}$ and $x_{2m+1}\in Sx_{2m}$ such that

$$\begin{aligned}
p\left(x_{2m+2},x_{2m+1}\right) &\leq H_p\left(Tx_{2m+1},Sx_{2m}\right)\\
&\leq \frac{1}{\sqrt{r}}\begin{bmatrix} ap\left(x_{2m+1},x_{2m}\right)\\ +b\left[p\left(x_{2m+1},Tx_{2m+1}\right)+p\left(x_{2m},Sx_{2m}\right)\right]\\ +c\left[p\left(x_{2m+1},Sx_{2m}\right)+p\left(x_{2m},Tx_{2m+1}\right)\right]\end{bmatrix}\\
&= \frac{1}{\sqrt{r}}\begin{bmatrix} ap\left(x_{2m+1},x_{2m+1}\right)\\ +b\left[p\left(x_{2m+1},x_{2m+2}\right)+p\left(x_{2m+1},x_{2m+1}\right)\right]\\ +c\left[p\left(x_{2m+1},x_{2m+1}\right)+p\left(x_{2m+1},x_{2m+2}\right)\right]\end{bmatrix}\\
&\leq rp\left(x_{2m+1},x_{2m+2}\right).
\end{aligned}$$

So, we obtain $(1-r)\,p\left(x_{2m+2},x_{2m+1}\right)\leq 0$. Since $r<1$, then $p\left(x_{2m+2},x_{2m+1}\right)=0$, which satisfy that $p\left(Tx_{2m+1},x_{2m+1}\right)=0$, and we get $x_{2m+1}\in Tx_{2m+1}$. As a result, $x_{2m}=x_{2m+1}$ is the common fixed point of S and T. A similar result obtain if $x_{2m+1}=x_{2m+2}$ for some $m\in\mathbb{N}^+$. Hence, we now consider that $x_m\neq x_{m+1}$ for all $m\in\mathbb{N}^+$. There are two cases.

If m is odd, from 24, we get

$$\begin{aligned}
H_p\left(Tx_{m+1},Sx_m\right) &\leq ap\left(x_{m+1},x_m\right)+b\left[p\left(x_{m+1},Tx_{m+1}\right)+p\left(x_m,Sx_m\right)\right]\\
&\quad +c\left[p\left(x_{m+1},Sx_m\right)+p\left(x_m,Tx_{m+1}\right)\right].
\end{aligned}$$

Hence, by Lemma 1, we have $x_{m+2}\in Tx_{m+1}$ and $x_{m+1}\in Sx_m$ such that

$$\begin{aligned}
p\left(x_{m+2},x_{m+1}\right) &\leq H_p\left(Tx_{m+1},Sx_m\right)\\
&\leq \frac{1}{\sqrt{r}}\begin{bmatrix} ap\left(x_{m+1},x_m\right)+b\left[p\left(x_{m+1},Tx_{m+1}\right)+p\left(x_m,Sx_m\right)\right]\\ +c\left[p\left(x_{m+1},Sx_m\right)+p\left(x_m,Tx_{m+1}\right)\right]\end{bmatrix}\\
&= \frac{1}{\sqrt{r}}\begin{bmatrix} ap\left(x_{m+1},x_m\right)+b\left[p\left(x_{m+1},x_{m+2}\right)+p\left(x_m,x_{m+1}\right)\right]\\ +c\left[p\left(x_{m+1},x_{m+1}\right)+p\left(x_{m+1},x_{m+2}\right)\right]\end{bmatrix}\\
&\leq \frac{1}{\sqrt{r}}\left[(a+2b+2c)\,p\left(x_{m+1},x_m\right)+(a+2b+2c)\,p\left(x_{m+1},x_{m+2}\right)\right]\\
&\leq \frac{\sqrt{r}}{1-\sqrt{r}}p\left(x_{m+1},x_m\right).
\end{aligned}\tag{25}$$

If m is even, we can obtain the same inequality, similarly. From here, we obtain that $\{p\left(x_k,x_{k+1}\right)\}$ is a positive and nonincreasing sequence of real numbers. So, from (25), we have

$$p\left(x_k,x_{k+1}\right)\leq r^k p\left(x_0,x_1\right)\tag{26}$$

for each $k\in\mathbb{N}$. Using (26) and the property (P4) of the partial metric, for any $m\in\mathbb{N}$, we get

$$\begin{aligned}
p\left(x_n,x_{n+m}\right) &\leq p\left(x_n,x_{n+1}\right)+p\left(x_{n+1},x_{n+2}\right)+\ldots+p\left(x_{n+m-1},x_{n+m}\right)\\
&\leq \left(\sqrt{r}\right)^n p\left(x_0,x_1\right)+\left(\sqrt{r}\right)^{n+1} p\left(x_0,x_1\right)+\ldots\\
&\quad +\left(\sqrt{r}\right)^{n+m-1} p\left(x_0,x_1\right)\\
&\leq \frac{\left(\sqrt{r}\right)^n}{1-\sqrt{r}}p\left(x_0,x_1\right)\to 0 \text{ as } n\to+\infty
\end{aligned}$$

since $0<r<1$. This shows that $\{x_n\}$ is a Cauchy sequence in (X,p^s). Since (X,p) is a complete partial metric space, by Lemma 1, (X,p^s) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to some $x\in X$ with respect to the metric p^s, that is, $\lim_{n\to\infty} p^s(x_n,x)=0$. Again from (1), we have

$$p(x,x)=\lim_{n\to\infty} p(x_n,x)=\lim_{n\to\infty} p(x_n,x_n)=0. \tag{27}$$

Now, we show that $x\in Tx$. Assume that $x\notin Tx$,namely $p(x,Tx)>0$. Due to (24), for a $\{x_{n+1}\}$ subsequence of $\{x_n\}$, we have

$$\begin{aligned} p(Tx,x) &\le p(Tx,x_{n+1})+p(x_{n+1},x)-p(x_{n+1},x_{n+1}) \\ &\le H_p(Tx,Sx_n)+p(x_{n+1},x) \\ &\le ap(x,x_n)+b[p(x,Tx)+p(x_n,Sx_n)] \\ &\quad +c[p(x,Sx_n)+p(x_n,Tx)]+p(x_{n+1},x) \end{aligned}$$

letting $n\to\infty$ and taking into account (27), the above inequality satisfy that

$$p(x,Tx)\le (b+c)\,p(x,Tx).$$

But this is impossible for $a,b,c\ge 0$ and $a+2a+2b<1$. Thus, $p(x,Tx)=0$. Therefore, from (27), we have

$$p(x,Tx)=0=p(x,x),$$

which from (2) implies that $x\in Tx$. Similarly, if we choose for a $\{x_{n+2}\}$ subsequence of $\{x_n\}$, we obtain $p(x,Sx)=0$. This completes the proof.

Further, following common fixed point theorem can be given for a contractive condition (28).

Theorem 6. *Suppose that (X,p) be a partial metric space and $S,T:X\to\Omega_C^p(X)$. If there exists $a+b+2c=r\in[0,1)$ such that*

$$\begin{aligned} H_p(Tx,Sy) \le\; & a\max\left\{p(x,y),p(x,Tx),p(y,Sy)\,\frac{p(x,Sy)+p(y,Tx)}{2}\right\} \\ & +b\max\{p(x,Tx),p(y,Sy)\} \\ & +c[p(x,Sy)+p(y,Tx)] \end{aligned} \tag{28}$$

for any $x,y\in X$. Then there exist $z\in X$ such that $z\in Tz$ and $z\in Sz$.

A similar proof verifies that T and S have a common fixed point z.

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