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On the Double Rational Chebyshev Functions: Definition, Properties and Application for Partial Differential Equations

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Abstract: In this paper, the concept of double rational Chebyshev (RC) functions on semi-infinite domain ($0 \leq x, y < \infty$) and some of their properties are introduced for the first time by the authors. Also, the definition of derivatives for double RC functions is deduced. The proposed definition is employed to deal with partial differential equations with variable coefficients, especially equations defined on semi-infinite domain, using the collocation method. The proposed technique is examined by some numerical test problems to investigate applicability, efficiency and accuracy. The obtained numerical results are compared with other existing methods and the exact solution where it is shown to be very attractive and maintains better accuracy. The technique seems to be very efficient, reliable, accurate and suitable to handle similar problems defined on infinite domains.

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1 Introduction

The spectral methods have an important and significant role in treating many phenomena and models in physics, engineering and many other fields. The choice of trial functions gives the spectral method a great distinguish feature, and depends on the exact solution of the differential equation $f(x)$ and the values of independent variable x. For example, if the solution $f(x)$ is polynomial in finite domain, we use Chebyshev polynomials, but if $f(x)$ is periodic, we prefer to use Fourier series. Recently many authors studied the application of Chebyshev polynomial and rational Chebyshev collocation method for solving different problems of differential, integro-differential equations and some other physical problems with variable coefficients. Ayşegül Akyüz and Mehmet Sezer [1] developed the use of Chebyshev collocation method for the solution of systems of high-order linear differential equations with variable coefficients. Elçin Gökmen and Mehmet Sezer [2] developed the use of Taylor collocation method for the solution of systems of high-order linear differential -difference equations with variable coefficients. Gamze Yüksel at el. [3] considered a Chebyshev polynomial approach for higher order linear Fredholm-Volterra integro-differential equations. Also, Chebyshev polynomials are applied to derive efficient algorithms for the solution of optimal control problems, see [4].

If $f(x)$ defined on the whole domain then the use of Hermit functions or exponential Chebyshev functions is optimal. For exponential Chebyshev functions, definitions and applications for solving ordinary and partial differential equations in unbounded domains, we refer the interested reader to the papers by Ramadan and Abd El Salam [5] and by Ramadan et al. [6-8]. In addition, if the solution is defined on semi-infinite interval $x \in [0, \infty)$ we use rational Chebyshev functions. Chebyshev polynomials are defined on square $(-1 \leq x < 1)$ in one variable but many of researchers have worked to extended them to multi-variable case, especially in two variables [9-12]. We noticed that if the differential equation is defined on large domain, especially when it is defined on unbounded domain, the use of Chebyshev approach causes a failure and weak approximation. For this reason, it is more suitable to generate a new basis set for the semi-infinite interval using a transformation that maps a finite domain into a semi-infinite interval, this idea is introduced by Boyed in 1987 [13] where, the new basis will get the most of the good numerical characteristics of the Chebyshev polynomials called

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RC functions that are orthogonal in $L_2(0,\infty)$. Ramadan et al. [14-19], Parand and Razzaghi [20], Parand et al. [21], and Sezer et al. [22] used rational Chebyshev function to solve differential equations and its applications. Also, RC functions are used to approximate model order reduction using harmony search, see [23].

In this work the definition of RC functions in two-variable is introduced, where this paper contains definition of double rational RC functions on semi-infinite domain ($0 \leq x, y < \infty$), and some properties of double RC functions are obtained. Also, the derivatives of RC functions in two variables are deduced in this work. The double RC collocation method is used for solving partial differential equations defined on the semi-infinite domains.

This paper is devoted to a computer aided analysis of subgroups of small index in Γ_d, in particular, we will deal as properties with subgroups of small index in Γ_d for $d = -1, -2, -3, -5, -7$. There is a beneficial interaction between geometric-topological, group theoretical and arithmetic questions and methods.

2 Double rational Chebyshev functions

The rational Chebyshev functions $R_n(x)$ of the first kind are functions in one variable x of degree n, defined by the relation

$$R_n(x) = T\left(\frac{x-1}{x+1}\right), when\ x = \cot^2(\theta/2),\ x \in [0,\infty). \tag{1}$$

2.1 *Definition of double RC functions*

The expression $T_{m,n}(x,y) = T_m(x)\,T_n(y)$ has been given by Basu [11] and for linear operator [24] and [25]. Now, we extend this definition for RC functions in double form as follows:

$$R_{m,n}(x,y) = R_m(x)R_n(y), where\ R_m(x) = T\left(\frac{x-1}{x+1}\right), R_n(y) = T\left(\frac{y-1}{y+1}\right). \tag{2}$$

2.2 *Properties of Double RC functions*

$$\begin{aligned} R_{r+1,s}(x,y) &= [2R_1(x)R_r(y) - R_{r-1}(x)]R_s(y), \\ R_{r,s+1}(x,y) &= R_r(x)[2R_1(y)R_s(y) - R_{s-1}(y)]. \end{aligned} \tag{3}$$

The weight function $w(x,y) = \frac{1}{(x+1)(y+1)\sqrt{xy}}$, is proper for choice for the double RC functions to be orthogonal. And we have

$$\int_0^\infty \int_0^\infty R_{r,s}(x,y)R_{m,n}(x,y)w(x,y)dxdy = \begin{cases} \pi^2 & r=s=m=n=0 \\ \frac{\pi^2}{4} & r=m\neq 0,\ s=n\neq 0 \\ \frac{\pi^2}{2} & r=m=0,\ s=n\neq 0 \\ & or\ r=m\neq 0,\ s=n=0 \\ 0 & otherwise \end{cases}.$$

Also, the multiplication relation of double RC functions is defined by

$$R_{r,s}(x,y)\cdot R_{m,n}(x,y) = \frac{1}{4}[R_{r+m,s+n}(x,y) + R_{r+m,|s-n|}(x,y) + R_{|r-m|,s+n}(x,y) + R_{|r-m|,|s-n|}(x,y)]. \tag{4}$$

2.3 *Using double RC functions to expand function*

A function $g(x,y)$ defined on semi-infinite interval $x,y \in [0,\infty)$, takes the form

$$g(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{m,n}R_{m,n}(x,y), \tag{5}$$

where

$$\lambda_{m,n} = \frac{\int_0^\infty \int_0^\infty g(x,y)R_{m,n}(x,y)w(x,y)dxdy}{\int_0^\infty \int_0^\infty R_{m,n}^2(x,y)w(x,y)dxdy}.$$

If $g(x,y)$ in (5) is truncated to $i,j < \infty$ it is expressed by the form

$$g(x,y) \cong \sum_{r=0}^{i}\sum_{s=0}^{j} \lambda_{r,s} R_{r,s}(x,y) = \mathrm{R}(x,y)\cdot \Lambda, \tag{6}$$

where $\mathrm{R}(x,y)$ is a $1\times(i+1)(j+1)$ row vector has components $R_{r,s}(x,y)$and Λ is a $1\times(i+1)(j+1)$ column vector required to be computed are expressed as

$$\begin{aligned}\mathrm{R}(\alpha,\beta) = [&R_{0,0}(x,y)\ \ R_{0,1}(x,y)\ \ \ldots.\ \ R_{0,j}(x,y)\ R_{1,0}(x,y)\ \ R_{1,1}(x,y)\ \ \ldots.\ \ R_{1,j}(x,y)\\ &\ldots..\ R_{i,0}(x,y)\ \ R_{i,1}(x,y)\ \ \ldots.\ \ R_{i,j}(x,y)],\end{aligned} \tag{7}$$

$$\Lambda = [\lambda_{0,0}\ \ \lambda_{0,1}\ \ \ldots.\ \ \lambda_{0,j}\ \ \lambda_{1,0}\ \ \lambda_{1,1}\ \ \ldots.\ \ \lambda_{1,j}\ \ldots..\ \lambda_{i,0}\ \ \lambda_{i,1}\ \ \ldots.\ \ \lambda_{i,j}]^T. \tag{8}$$

3 The partial derivatives of double RC functions

In this proposition we introduce the concept of partial derivatives of double RC functions in terms of itself.
Proposition 3.1
The derivative of double RC functions of order $(r+s)$th is given in terms of itself by the relation

$$\mathrm{R}^{(r,s)}(x,y) \cong \mathrm{R}(x,y)(\mathrm{D}_x)^r(\mathrm{D}_y)^s, \tag{9}$$

where, D_x and D_y are $(i+1)(j+1)\times(i+1)(j+1)$ which can be obtained by $\mathrm{D}_x = \mathrm{D}_1 + \mathrm{D}_2$,
where

$$\left.\begin{aligned}&\mathrm{D}_1 = diag\left(\tfrac{7}{4}rI,\ -rI,\ \tfrac{1}{4}rI\right), \quad r=0,\ 1,\ \ldots,\ i\\ &\mathrm{D}_2 = d_{r,s}I = \begin{cases}0 & s\geq r\\ lrc_sI & s<r\end{cases} \quad \text{and} \quad d_{21}I = -I, s=0,\ 1,\ \ldots,\ j\end{aligned}\right\}, \tag{10}$$

where $l = (-1)^{r+s+1}, c_1 = 1$ and $c_s = 2$ for $s \geq 2$,
and

$$\mathrm{D}_y = \begin{bmatrix}\omega & 0 & \cdots & 0\\ 0 & \omega & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \omega\end{bmatrix}^T, \quad \omega = S_1 + S_2, and\ s_{21} = -1, \tag{11}$$

where,
$S_1 = diag.\left(\tfrac{7}{4}s,\ -s,\ \tfrac{1}{4}s\right)$, s=0,1, 2, …, i,
and the s_{rs} components of S_2 are obtained from:

$$s_{rs} = \begin{cases}0, & s\geq r\\ l(r)c_s, & s<r\end{cases},$$

where I is identity matrix , 0 is zero matrix and ω is matrix. The dimensions of these three matrices are $(j+1)\times(j+1)$ which are components of the matrices D_x and D_y.
Proof
By using recurrence relation (3) we can demonstrate partial derivatives of the double RC functions, first dealing with the variable x, and by using the multiplication relation we get

$$\frac{\partial}{\partial x}\left(R_{0,n}(x,y)\right) = 0, \tag{12}$$

for all n

$$\begin{aligned}\tfrac{\partial}{\partial x}\left(R_{1,n}(x,y)\right) &= \tfrac{4}{(1+x)^2}R_n(y) = \left(\tfrac{3}{4}R_0(x) - R_1(x) + \tfrac{1}{4}R_2(x)\right)R_n(y)\\ &= \tfrac{3}{4}R_{0,n}(x,y) - R_{1,n}(x,y) + \tfrac{1}{4}R_{2,n}(x,y),\end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\left(R_{m+1,n}(x,y)\right) &= \frac{\partial}{\partial x}[2R_1(x)R_{m,n}(x,y) - R_{m-1,n}(x,y)],\\ &= \frac{\partial}{\partial x}[2\left(R_1(x)\right)^{(0,0)}\left(R_{m,n}(x,y)\right)^{(0,0)} - \left(R_{m-1,n}(x,y)\right)^{(0,0)}],\\ &= [2\left(R_1(x)\right)^{(1,0)}\left(R_{m,n}(x,y)\right)^{(0,0)} + 2\left(R_1(x)\right)^{(0,0)}\left(R_{m,n}(x,y)\right)^{(1,0)} - \left(R_{m-1,n}(x,y)\right)^{(1,0)}].\end{aligned} \tag{14}$$

Using the relations (12), (13) and (14) and by using multiplication relation (4) for m=0, 1, ..., i, the components of the block matrix of derivatives D_x can be shown as following:

$$\begin{cases} \left(R_{0,n}(x,y)\right)^{(1,0)} = 0,\\ \left(R_{1,n}(x,y)\right)^{(1,0)} = \frac{3}{4}R_{0,n}(x,y) - R_{1,n}(x,y) + \frac{1}{4}R_{2,n}(x,y),\\ \left(R_{2,n}(x,y)\right)^{(1,0)} = -2R_{0,n}(x,y) + \frac{7}{2}R_{1,n}(x,y) - 2R_{2,n}(x,y) + \frac{1}{2}R_{3,n}(x,y),\\ \left(R_{3,n}(x,y)\right)^{(1,0)} = 3R_{0,n}(x,y) - 6R_{1,n}(x,y) + \frac{21}{4}R_{2,n}(x,y) - 3R_{3,n}(x,y) + \frac{3}{4}R_{4,n}(x,y),\\ \quad\vdots \end{cases} \tag{15}$$

Similar to partial derivative with respect to x, we can show partial derivative with respect to y, hence we write the components of the matrix of derivatives D_y in the form

$$\begin{cases} \left(R_{m,0}(x,y)\right)^{(1,0)} = 0,\\ \left(R_{m,1}(x,y)\right)^{(1,0)} = \frac{3}{4}R_{m,0}(x,y) - R_{m,1}(x,y) + \frac{1}{4}R_{m,2}(x,y),\\ \left(R_{m,2}(x,y)\right)^{(1,0)} = -2R_{m,0}(x,y) + \frac{7}{2}R_{m,1}(x,y) - 2R_{m,2}(x,y) + \frac{1}{2}R_{m,3}(x,y),\\ \left(R_{m,3}(x,y)\right)^{(1,0)} = 3R_{m,0}(x,y) - 6R_{m,1}(x,y) + \frac{21}{4}R_{m,2}(x,y) - 3R_{m,3}(x,y) + \frac{3}{4}R_{m,4}(x,y),\\ \quad\vdots \end{cases} \tag{16}$$

We assume:
$\left(R_{m,n}(x,y)\right)^{(1,0)} = \left(R_{m,n}(x,y)\right)^{(0,1)} = \left(R_{m,n}(x,y)\right)^{(0,0)} = 0$ for $m > i$ and $n > j$.
This assumption is based on truncating the matrices D_x and D_y to be square and the matrix multiplication become possible. Thus, to find $\mathrm{R}^{(r,s)}$ (x,y) using the equalities (15), (16) as

$$\mathrm{R}^{(1,0)}(x,y) = \mathrm{R}(x,y)\mathrm{D}_x,$$

$$\mathrm{R}^{(2,0)}(x,y) = \mathrm{R}^{(1,0)}(x,y)\mathrm{D}_x = \left(\mathrm{R}(x,y)\mathrm{D}_x\right)\mathrm{D}_x = \mathrm{R}(x,y)\left(\mathrm{D}_x\right)^2,$$
$$\mathrm{R}^{(3,0)}(x,y) = \mathrm{R}^{(2,0)}(x,y)\left(\mathrm{D}_x\right)^2 = \mathrm{R}(x,y)\left(\mathrm{D}_x\right)^3.$$

Then,

$$\mathrm{R}^{(r,0)}(x,y) = \mathrm{R}(x,y)\left(\mathrm{D}_x\right)^r. \tag{17}$$

Also,

$$\mathrm{R}^{(0,s)}(x,y) = \mathrm{R}(x,y)\left(\mathrm{D}_x\right)^s. \tag{18}$$

Hence, from (17) and (18) we will have

$$\mathrm{R}^{(r,s)}(x,y) \cong \mathrm{R}(x,y)\left(\mathrm{D}_x\right)^r\left(\mathrm{D}_y\right)^s. \tag{19}$$

4 Application for the double RC functions

To test the proposed definition we will use double RC collocation method to solve partial differential equations defined on semi-infinite domain which takes the form [26]

$$\sum_{r=0}^{i}\sum_{s=0}^{j} p_{r,s}(x,y)\,\mu^{(r,s)}(x,y) = q(x,y), 0 \le (x,y) < \infty, \tag{20}$$

with the nonlocal conditions

$$\sum_{t=1}^{v}\sum_{r=0}^{i}\sum_{s=0}^{j} b_{r,s}^{t}\mu^{(r,s)}\left(\phi_t,\varphi_t\right) = \kappa,$$

or

$$\sum_{t=1}^{v}\sum_{r=0}^{i}\sum_{s=0}^{j} c^{t}_{r,s}(x)\mu^{(r,s)}(\alpha,\theta_t) = u(x), \tag{21}$$

or

$$\sum_{t=1}^{v}\sum_{r=0}^{i}\sum_{s=0}^{j} d^{t}_{r,s}(y)\,\mu^{(r,s)}(\varepsilon_t,y) = h(y),$$

where the $\mu^{(0,0)}(x,y)$=$\mu(x,y)$, $\mu^{(r,s)}(x,y) = \frac{\partial^{r+s}}{\partial x^r \partial y^s}\mu(x,y)$and $p_{r,s}(x,y)$, $q(x,y)$, $c^t_{r,s}(x)$, $u(x)$, $d^k_{ij}(y)$ and $h(y)$are defined on semi-infinite interval $[0,\infty)$ and $\phi_t, \varphi_t, \theta_t, \varepsilon_t$ are invariable $\in [0,\infty)$,especial case if one or more of them tend to infinity. We suppose that the approximate solution $A(x,y)$ to the exact solution $\mu(x,y)$ of Eq. (20) from (6) and its (r, s)th order derivatives deduced in Eq. (19) are represented as

$$A(x,y) = \sum_{r=0}^{i}\sum_{s=0}^{j}\lambda_{r,s}R_{r,s}(x,y) = \mathrm{R}(x,y)\cdot\Lambda, \tag{22}$$

and

$$A^{(r,s)}(x,y) = [\mathrm{R}(x,y)(\mathrm{D}_x)^r(\mathrm{D}_x)^s]\Lambda. \tag{23}$$

The collocation points x_τ and y_η will defined in the following:

$$x_\tau = \left(\frac{1+\cos\left(\frac{\tau\pi}{i}\right)}{1-\cos\left(\frac{\tau\pi}{i}\right)}\right), \quad y_\eta = \left(\frac{1+\cos\left(\frac{\eta\pi}{j}\right)}{1-\cos\left(\frac{\eta\pi}{j}\right)}\right), \quad (\tau = 1,\ \dots,\ i-1,\ \eta = 1,\ \dots,\ j-1), \tag{24}$$

where $0 \le x_\tau, y_\eta < \infty$.

If the variables x and y defined in finite interval $x \in [0,a]$ and $y \in [0,b]$where a and b are any positive numbers we prefer using another collocation points

$$x_\tau = \frac{c}{m}\tau,\ \tau = 0,1,\dots,m, y_\eta = \frac{c}{n}\eta,\ \eta = 0,1,\dots,n.$$

The double RC functions are specified by convergent to x and y even if they tend to infinity, for this, doesn't make failure in the method in unbounded domain. Then, substituting the collocation points (24) into (20) we get

$$\sum_{r=0}^{i}\sum_{s=0}^{j} p_{r,s}(x_\tau,y_\eta)\,\mu^{(r,s)}(x_\tau,y_\eta) = q(x_\tau,y_\eta), \tag{25}$$

or briefly by using matrix form

$$\sum_{r=0}^{i}\sum_{s=0}^{j}\mathrm{P}_{r,s}\,\mathrm{A}^{(r,s)} = \mathrm{Q}, \tag{26}$$

where $\mathrm{P}_{r,s}$ is the block matrix which has the diagonal elements $p_{r,s}(x_\tau,y_\eta)$and the other elements are zeros and $\boldsymbol{Q}$ is the block vector with the components $q(x_\tau,y_\eta)$. Bulging the collocation points (24) into partial derivatives of approximate solution $A(x,y)$ we get

$$\mathrm{A}^{(r,s)} = \begin{bmatrix} A^{(r,s)}(x_0,y_0) \\ \vdots \\ A^{(r,s)}(x_0,y_j) \\ A^{(r,s)}(x_1,y_0) \\ \vdots \\ A^{(r,s)}(x_1,y_j) \\ \vdots \\ A^{(r,s)}(x_i,y_j) \end{bmatrix} = [\mathrm{R}(\mathrm{D}_x)^r(\mathrm{D}_y)^s]\Lambda, \tag{27}$$

where
$$\mathrm{R}=[\mathrm{R}(x_0,y_0)\ \ \mathrm{R}(x_0,y_1)\ \ \dots\ \ \mathrm{R}(x_0,y_j)\ \mathrm{R}(x_1,y_0)\ \ \mathrm{R}(x_1,y_1)\ \ \dots\ \ \mathrm{R}(x_1,y_j)\ \ \dots\ \ \mathrm{R}(x_i,y_0)\ \ \dots\ \mathrm{R}(x_i,y_1)\ \ \mathrm{R}(x_i,y_2)\ \ \dots\ \ \mathrm{R}(x_i,y_j)]^T.$$

Therefore, from Eq. (27), we get a system of the form

$$\left(\sum_{r=0}^{i}\sum_{s=0}^{j}\mathrm{P}_{r,s}\left\{\mathrm{R}(\mathrm{D}_x)^r(\mathrm{D}_y)^s\right\}\right)\Lambda=\mathrm{Q},\tag{28}$$

which is an $(i+1)\times(j+1)$system of linear algebraic equations with $(i+1)\times(j+1)$ $\lambda_{r,s}$unknowns. Substituting the collocation points (24) in the condition (21) we get the fundamental matrices of the form

$$\begin{gathered}\sum_{t=1}^{v}\sum_{r=0}^{i}\sum_{s=0}^{j}b_{r,s}^{t}\left\{\mathrm{R}\left(\phi_t,\varphi_t\right)(\mathrm{D}_x)^r(\mathrm{D}_y)^s\right\}\Lambda=\kappa,\\ \sum_{t=1}^{v}\sum_{r=0}^{i}\sum_{s=0}^{j}c_{r,s}^{t}(x_\tau)\left\{\mathrm{R}\left(\alpha_\tau,\theta_t\right)(\mathrm{D}_x)^r(\mathrm{D}_y)^s\right\}\Lambda=u(x_\tau),\\ \sum_{t=1}^{v}\sum_{r=0}^{i}\sum_{s=0}^{s}d_{r,s}^{t}(y_\eta)\left\{\mathrm{R}\left(\varepsilon_t,y_\eta\right)(\mathrm{D}_x)^r(\mathrm{D}_y)^s\right\}\Lambda=h(y_\eta).\end{gathered}\tag{29}$$

5 Method of Solution

The fundamental matrix (28) for Eq. (20) is transformed to a system of $(i+1)\times(j+1)$ algebraic equations for the $(i+1)\times(j+1)$ unknown RC coefficients

$$[\lambda_{0,0},\ \lambda_{0,1},\ \dots\ \lambda_{0,j},\ \lambda_{1,0},\ \lambda_{1,1},\ \dots\ \lambda_{1,j},\ \dots.\ ,\lambda_{i,0},\ \lambda_{i,1},\ \dots\ \lambda_{i,j}].$$

We can write the matrix (27) as:

$$\boldsymbol{Y\Lambda}\textbf{=}\boldsymbol{Q}\ or\ [\boldsymbol{Y};\boldsymbol{Q}].\tag{30}$$

Also, we write the fundamental matrix of the conditions from (29) as follows

$$\mathrm{X}\Lambda=\mathrm{G}\quad or\ [\mathrm{X};\mathrm{G}]\quad .\tag{31}$$

where $\boldsymbol{X}$ is a $h\times(i+1)(j+1)$matrix and $\boldsymbol{G}$ is a $h\times 1$matrix, so that h is the rank of the all row matrices as in (30) belong to the given conditions.
Thus, systems (30) and (31) can be expressed as follows:

$$\mathrm{Y}^*\Lambda=\mathrm{Q}^*\ or\ [\mathrm{Y}^*;\mathrm{Q}^*].\tag{32}$$

Hence, the equations (32) can be compacted by putting the vectors (31) on conditions to the equations (30). We use the generalized inverse [27] of $\mathrm{Y}^*$for solving equations (32), then we get the unknown $\lambda_{r,s}$from the following:
$\Lambda=\mathrm{geninv}(\mathrm{Y}^*)\cdot\mathrm{Q}^*$.

6 Numerical Examples

In this section, we apply the proposed technique on some test examples to illustrate its applicability and validity. All numerical examples are computed on the computer by MATHEMATICA 7.0.
Problem 1
Consider the partial differential equation of order two

$$\mu^{(0,2)}(x,y)+\mu^{(1,1)}(x,y)=\frac{4y-4x^2+8}{(x+1)^2(y+1)^3},\quad x,y\in[0,\infty),$$

and the conditions for this test example are $\mu(x,y) = 1$ at $x \to \infty$ and at $y \to \infty, \mu(0,0) = 1$, the fundamental matrix takes the form

$$\left\{ \mathrm{R}(\mathrm{D}_y)^2 + \mathrm{R}\mathrm{D}_x\mathrm{D}_y \right\} \Lambda = \mathrm{Q},$$

consider *i=j=4*, where, the approximate solution has the form

$$A(x,y) = \lambda_{0,0}R_{0,0}(x,y) + \lambda_{0,1}R_{0,1}(x,y) + \cdots + \lambda_{4,4}R_{4,4}(x,y).$$

Then, after the augmented matrix of the system and conditions are computed, we obtain the solution as,

$$\lambda_{0,0} = \ \lambda_{0,1} = \ \ldots \ = \lambda_{0,4} = 0, \lambda_{1,0} = 0, \ \lambda_{1,1} = 1, \lambda_{1,2} = 0, \ldots . = \lambda_{1,4} = 0, \lambda_{2,0} = 0, \ \lambda_{2,1} = 0, \ \ldots \ = \lambda_{2,4} = 0,$$
$$\lambda_{3,0} = 0, \ \lambda_{3,1} = 0, \ \ldots \ = \lambda_{3,4} = 0, \lambda_{4,0} = \ \lambda_{4,1} = \ \ldots = \ \lambda_{4,4} = 0.$$

Then, using the relation (2) we get $\mu(x,y) = \frac{xy-x-y+1}{(x+1)(y+1)}$, which is the exact solution.

Problem 2

Consider Piossion equation

$$\mu^{(2,0)}(x,y) + \mu^{(0,2)}(x,y) = -2e^{-x-y}, \quad x,y \in [0,1],$$

with the Dirichlet boundary conditions

$$\mu(0,y) = e^{-y}, \quad \mu(x,0) = e^{-x} \quad \mu(1,y) = e^{-1-y}, \quad \mu(x,1) = e^{-x-1}.$$

We apply the introduced technique and solve this example. It has seen from Table 1 and 2 the numerical results for i=j = 8, 10 and 15 comparing with the exact solution which is $\mu(\alpha,\beta) = e^{-\alpha-\beta}$ are tabulated. In addition, the absolute error functions for different values of i and j are illustrated in Figure 1, Figure 2 and Figure 3. Table 3 shows the CPU time of the proposed the time is mentioned by seconds.

Table. 1 absolute errors for various values of i, j where y= 0.1 and x takes various values

x	Exact solution	Present method i=j=8	Absolute error	Present method i=j=10	Absolute error	Present method i=j=15	Absolute error
0	0.9048374180	0.9048371375	2.804×10^{-7}	0.9048374246	6.626×10^{-9}	0.9048374521	3.4074×10^{-8}
0.1	0.8187307530	0.8187312589	5.058×10^{-7}	0.8187342425	3.489×10^{-6}	0.8187302764	4.76599×10^{-7}
0.2	0.7408182206	0.7408204798	2.259×10^{-6}	0.7408218443	3.623×10^{-6}	0.7408174809	7.39759×10^{-7}
0.3	0.6703200460	0.6703153776	4.668×10^{-6}	0.6703182998	1.746×10^{-6}	0.6703191313	9.14714×10^{-7}
0.4	0.6065306597	0.6065197858	1.087×10^{-5}	0.6065377883	7.128×10^{-6}	0.6065296425	1.0172×10^{-6}
0.5	0.5488011924	0.5488011924	1.044×10^{-5}	0.5488308946	1.925×10^{-8}	0.5488105886	1.04743×10^{-6}
0.6	0.4965853037	0.4965803342	4.96×10^{-6}	0.4966083685	2.306×10^{-5}	0.4965842797	1.02401×10^{-6}
0.7	0.4493289641	0.4493302679	1.303×10^{-6}	0.449346711	1.774×10^{-5}	0.4493280322	9.31863×10^{-7}
0.8	0.4065696597	0.4065746321	4.972×10^{-6}	0.4065783577	8.697×10^{-6}	0.40656891160	7.48137×10^{-7}
0.9	0.3678794411	0.3678840147	4.573×10^{-6}	0.3678811024	3.74×10^{-6}	0.36787899285	4.48318×10^{-7}
1	0.3328710836	0.3328715031	4.194×10^{-7}	0.3328710407	4.293×10^{-8}	0.33287109707	1.33749×10^{-8}

Table. 2 comparing the L_2 error norms and L_∞

	L_2	L_∞
i=j=8	9.57079×10^{-9}	1.29247×10^{-5}
i=j=10	4.37445×10^{-8}	1.77424×10^{-5}
i=j=*15*	1.37453×10^{-10}	1.17311×10^{-6}

Table.3 Comparing the CPU time (seconds)

	Proposed Method time
i=j=*8*	7.785
i=j=*10*	14.946
i=j=*15*	21.2356

Problem 3

Consider the following PDE

$$\mu_x - \mu_y = Cosx + Cosy + xSiny + ySinx, \quad x, y \in [0,1],$$

which has initial conditions $\mu(0,y) = -y,\ \mu(x,0) = x$.

We apply double RC collocation method for approximating the solution. The numerical results are obtained in Table 4 and 5 for values of i *and* j, using the previous technique comparing with the exact solution of $\mu(x,y) = x\,Cos\,y - y\,Cos\,x$ are tabulated with randomly chosen x and y. Also, we plot figures 4, 5 and 6 to obtain the absolute errors for different values of i, j. In Table 6 comparison of the L_2 and L_∞ error norms of present method are presented with different values of i, j. Table 7 shows the CPU time of the proposed the time is mentioned by seconds.

Table.4 Comparing approximate and exact solution

x	y	exact	*i=j*=8	*i=j* =9	*i=j* =10
0.0	0.3	-0.3	-0.299991	-0.300001	-0.300004
0.1	0.7	-0.620019	-0.620023	-0.619986	-0.619987
0.2	0.9	-0.757738	-0.757643	-0.757865	-0.757718
0.4	0.2	0.207814	0.207768	0.207886	0.2078000
0.5	0.7	-0.231887	-0.232106	-0.231636	-0.231833
0.6	0.5	0.113882	0.113946	0.11375	0.113901
0.7	0.1	0.620019	0.620023	0.619982	0.61999
0.9	0.8	0.129748	0.129852	0.130003	0.129406

Table.5 Comparing Absolute error at different *i, j*

x	y	Absolute Error		
		i=j =8	*i=j* =9	*i=j* =10
0.0	0.3	9.39367×10^{-6}	1.09131×10^{-6}	4.07844×10^{-6}
0.1	0.7	4.33313×10^{-6}	3.29165×10^{-5}	3.20975×10^{-5}
0.2	0.9	9.46117×10^{-5}	1.27032×10^{-4}	6.40843×10^{-5}
0.4	0.2	4.63359×10^{-5}	7.1439×10^{-5}	1.40171×10^{-5}
0.5	0.7	2.19264×10^{-4}	2.50232×10^{-4}	5.37054×10^{-5}
0.6	0.5	6.40402×10^{-5}	1.32096×10^{-4}	1.96967×10^{-5}
0.7	0.1	4.41116×10^{-6}	3.62481×10^{-5}	1.02346×10^{-3}
0.9	0.8	1.04312×10^{-4}	2.55057×10^{-4}	3.42336×10^{-4}

Table.6 comparing the L_2 error norms and L_∞

	L_2	L_∞
i=j =8	1.52782×10^{-6}	1.29247×10^{-6}
i=j =9	1.9174×10^{-6}	1.77424×10^{-6}
i=j =10	3.41043×10^{-6}	1.17311×10^{-6}

Table.7 Comparing the CPU time (seconds)

	Proposed Method time
i=j=8	3.214
i=j=9	3.91
i=j=10	4.31

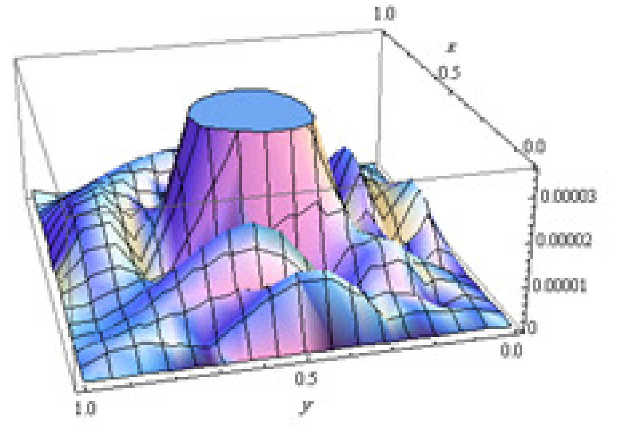

Figure.1 Error function at i=j =8

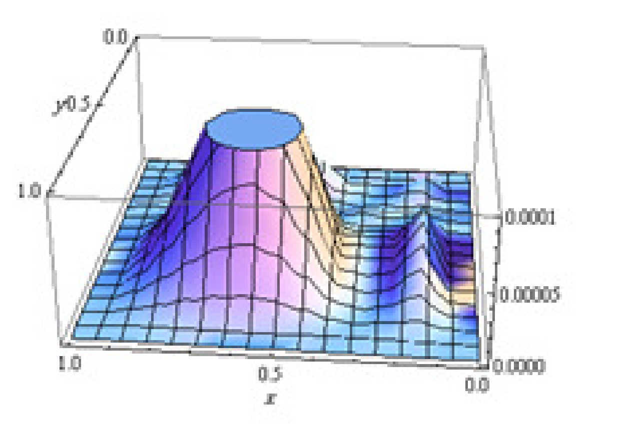

Figure.2 Error function at i=j =10

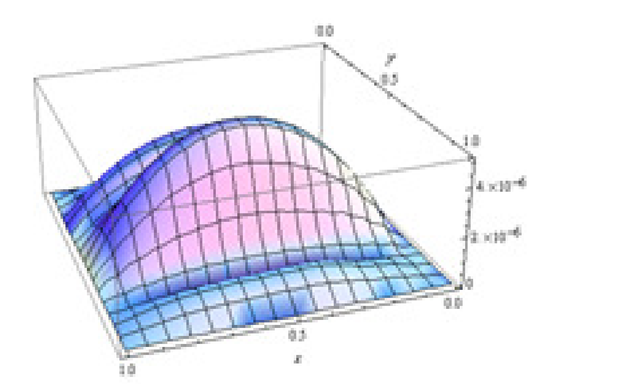

Figure.3 Error function at i=j =15

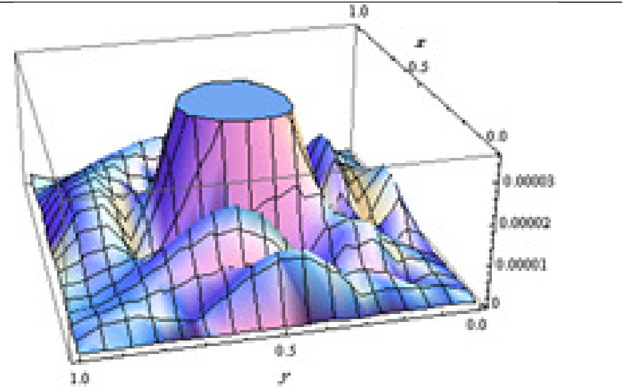

Figure.4 Error function at i=j =8

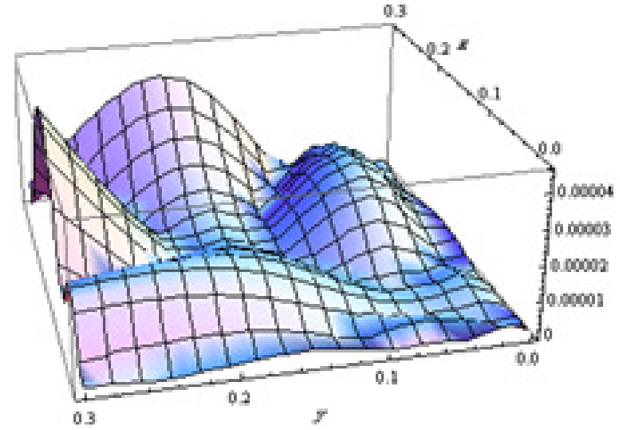

Figure.5 Error function at i=j =9

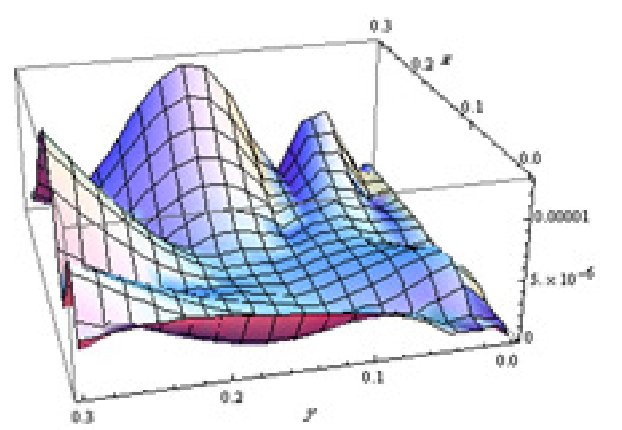

Figure.6 Error function at i=j =10

7 Conclusion

In this work, a collocation technique for high-order linear partial differential equations with variable coefficients under conditions is proposed. The technique is based on the approximating the solution function by the truncated double rational Chebyshev series. The definitions of the partial derivatives of double rational Chebyshev are explored. The PDEs and conditions are first transformed to matrix equations. This matrix equation is a system of linear algebraic equations with the unknown rational Chebyshev coefficients. Test examples are used to demonstrate the applicability, effectiveness and the accuracy of the proposed technique.

References

[1] Ayşegül Akyüz, Mehmet Sezer, Chebyshev polynomial solutions of systems of high-order linear differential equations with variable coefficients, Applied Mathematics and Computation 144 (2003) 237–247.

[2] Elçin Gökmen, Mehmet Sezer, Taylor collocation method for systems of high-order linear differential–difference equations with variable coefficients, Ain Shams Engineering Journal,2013 4, 117–125.

[3] G. Yuksel, M. Gulsu and M. Sezer, A Chebyshev Polynomial Approach for Higher – Order Linear Fredholm-Volterra Integro-Differential Equations, Gazi University Journal of Science, 2012 ,25(2), 393 – 401.

[4] B. Kafasha,, A. Delavarkhalafi , S.M. Karbassi, Application of Chebyshev polynomials to derive efficient algorithms for the solution of optimal control problems, Scientia Iranica ,Vol. 19, Issue 3, June 2012, Pages 795-805.

[5] Ramadan M.A. and Abd El Salam M. A., Solving systems of ordinary differential equations in unbounded domains by exponential Chebyshev collocation method, Journal of Abstract and Computational Mathematics,2016, No. 1, 33-46.

[6] Ramadan M.A., Raslan K.R., El- Danaf T.S. and El Salam M.A., An exponential Chebyshev second kind approximation for solving high – order ordinary differential equations in unbounded domains with application to Dawson integral , Journal of the Egyptian Mathematical Society,2016, 1-9.

[7] Ramadan M.A., Raslan K.R., El- Danaf T.S. and El Salam M.A, Modified exponential Chebyshev operational matrices of derivatives for solving high-order partial differential equations in unbounded domains, Electronic Journal of Mathematical Analysis and Applications, 2017,Vol. 5(2), July ,229-241.

[8] Ramadan M.A., Raslan K.R., El- Danaf T.S. and El Salam M.A**,** Solving systems of high-order ordinary differential equations with variable coefficients by exponential Chebyshev collocation method, Journal of Modern Methods in Numerical Mathematics ,2017,8:1-2, 40–51.

[9] Fox L, Parker IB. Chebyshev polynomials in numerical analysis. *Oxford University Press London* 1968.

[10] Mason JC, Handscomb DC. Chebyshev polynomials. *Chapman and Hall/CRC*; 2002.

[11] Basu NK. On double Chebyshev series approximation. *SIAM Journal on Numerical Analysis*. 1973 Jun;10(3):496-505.

[12] Doha EH. The Chebyshev coefficients of general-order derivatives of an infinitely differentiable function in two or three variables. *Ann. Univ. Sci. Budapest. Sect. Comput*. 1992;13:83-91.

[13] Boyd JP. Orthogonal rational functions on a semi-infinite interval. *J. Comput. Phys*.70:1,63- 88, (1987).

[14] Ramadan MA, Raslan KR, Nassar MA. An approximate analytical solution of higher-order linear differential equations with variable coefficients using improved rational Chebyshev collocation method. Applied and Computational Mathematics. 2014;3(6):315-22.

[15] Ramadan MA, Raslan KR, Nassar MA. A rational Chebyshev functions approach for Fredholm-Volterra integro-differential equations. *Computational Methods for Differential Equations*. 2015 Oct 1;3(4):284-97.

[16] Ramadan MA, Raslan KR, Nassar MA. An approximate solution of systems of high-order linear differential equations with variable coefficients by means of a rational Chebyshev collocation method. *Electron. J. Math. Anal. Appl*. 2016 Jan;4(1):86-98.

[17] Ramadan MA, Raslan KR, Nassar MA. Numerical solution of system of higher order linear ordinary differential equations with variable coefficients using two proposed schemes for rational Chebyshev functions. Global Journal of Mathematics. 2015 Aug 15;3(2):322-37.

[18] Ramadan MA, Raslan KR, Hadhoud AM, Nassar MA. Numerical solution of high-order linear integro-differential equations with variable coefficients using two proposed schemes for rational Chebyshev functions. *New trends in mathematical sciences.* 2016 ,Jun 1;4(3):22.

[19] Ramadan MA, Raslan KR, Hadhoud AM, Nassar MA. Rational Chebyshev functions with new collocation points in semi-infinite domains for solving higher-order linear ordinary differential equations. *Journal of advances in mathematics*. 2015 Nov 20;11(7):5403-5410.

[20] Parand K, Razzaghi M. Rational Chebyshev tau method for solving Volterra's population model. *Applied Mathematics and Computation*. 2004 Feb 22;149(3):893-900.

[21] Parand K, Delafkar Z, Baharifard F. Rational Chebyshev Tau method for solving natural convection of Darcian fluid about a vertical full cone embedded in porous media whit a prescribed wall temperature. *World Academy of Science, Engineering and Technology*,2011 Aug. 24;5(8):1186-1191.

[22] Sezer M, Gülsu M, Tanay B. Rational Chebyshev collocation method for solving higher-order linear ordinary differential equations. *Numerical Methods for Partial Differential Equations*. 2011 Sep;27(5):1130-42.

[23] H. Nasiri Soloklo, M. Maghfoori Farsangi , Chebyshev rational functions approximation for model order reduction using harmony search, Scientia Iranica, 2013, 20(3),771-777.

[24] Acu, A. M., Hodiş, S., & Raşa, I. (2018). A survey on estimates for the differences of positive linear operators. Constr. Math. Anal, 1(2), 113-127.

[25] Altomare, F. Iterates of Markov Operators and Constructive Approximation of Semigroups. Constructive Mathematical Analysis, 2(1), 22-39.

[26] Koç AB, Kurnaz A. A new kind of double Chebyshev polynomial approximation on unbounded domains. *Boundary value problems*. 2013 Dec 1;2013(1):10.

[27] Akyüz Dascioglu A. Chebyshev polynomial approximation for high order partial differential equations with complicated conditions. Numerical Methods for Partial Differential Equations: An International Journal. 2009 May;25(3):610-21.