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New fractional derivatives in the sense of Riemann-Liouville and Caputo approaches

Gustavo Asumu MBoro Nchama

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Abstract: In this paper we suggest new fractional derivatives which, from the theoretical viewpoint, improve the Riemann-Liouville and Caputo fractional derivatives. One of the derivatives is based upon the Riemann-Liouville view- point and the other one on the Caputo approach. Some properties are given to illustrate the results.

Keywords: Caputo fractional derivative, fractional calculus, Riemann-Liouville fractional derivative.

1 Introduction

A fractional derivative is an integral operator which generalizes the ordinary derivative, such that if the fractional derivative is represented by D^{α} then, when $\alpha =$, it coincides with the usual differential operator D^n [1]. Its origin dates back to 1695 when L'Hopital raised by a letter to Leibniz the question of how the expression

$$D^{\alpha}u(t) = \frac{d^n}{dt^n}u(t),$$

should be understood if n was a real number [2]. Since then, this new branch turned out to be very attractive to mathematicians such as Euler, Laplace, Fourier, Liouville, Riemann, Laurent, Weyl and Abel who first applied it in physics to solve the integral equation arising from the tautochron problem [3]. There are some specific spaces in which fractional derivatives are defined:

Definition 1. *A function $f:[a,b]\to\mathbb{R}$ is said to be absolutely continuous, denoted by $f\in AC[a,b]$ on $[a,b]$ if given $\varepsilon>0$, there exists some $\sigma>0$ such that*

$$\sum_k |f(y_k)-f(x_k)| < \varepsilon.$$

whenever $\{[x_k,y_k]: k=1,\cdots,n\}$ is a finite collection of mutually disjoint subintervals of $[a,b]$ with

$$\sum_k (y_k-x_k) < \sigma.$$

Definition 2. *Let $n\in\mathbb{N}$ and $k=1,2,\cdots,n-1$, the space $AC^n[a,b]$ is defined by*

$$AC^n[a,b] := \{f:[a,b]\to\mathbb{R} : f^{(k)}(t)\in C[a,b], f^{(n-1)}(t)\in AC[a,b]\}.$$

There are many types of fractional derivatives (three popular definitions were given by Gronwald-Letnikov, Riemann-Liouville and Caputo) [4-6]. One of the most used fractional derivatives was defined by Gronwald and Letnikov:

* Corresponding author e-mail: becquerrr10@hotmail.com

Definition 3. *Let* $a,b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in C^n[a,b]$. *The Gronwald-Letnikov fractional derivative of order* α*, is given by*

$$ {}^{GL}D_{at}^{\alpha} f(t) = \lim_{h \to 0^+} \frac{1}{h^\alpha} \sum_{k=0}^{\left[\frac{x-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(t - kh). \tag{1} $$

By induction, if f(t) has n continuous derivatives, it is proved in [7] that the formula (1) is equivalent to the following integral form

$$ {}^{GL}D_{at}^{\alpha} f(t) = \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a) + \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-(\alpha+1)} f^{(n)}(\tau) d\tau. \tag{2} $$

The relation (2) presents two inconveniences: on one hand, from the theoretical point of view, the class of functions for which this derivative is defined (n times continuously differentiable functions) is very narrow. On the other hand, the presence of non-integral terms in the right-hand-side of (2) does not look better. To overcome both inconveniences, Riemann and Liouville proposed the following definition [8]:

Definition 4. *Let* $a,b \in \mathbb{R}$, $a < b$, $n-1 < \alpha \leq n \in \mathbb{N}$, $f \in AC^n[a,b]$. *The Riemann-Liouville fractional derivative of order* α*, is defined by*

$$ {}^{RL}D_{at}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-(\alpha+1)} f(s) ds. $$

Even though the Riemann-Liouville approach overcomes the drawbacks related the Gronwald-Letnikov definition and it has been applied successfully in many areas of engineering, unfortunately, it leads to initial conditions containing the limit values of the Riemann-Liouville fractional derivative at the lower terminal t = a, for example

$$ \begin{aligned} &\lim_{t \to a} \{ {}^{RL}D_{at}^{\alpha-1} f(t) \} = b_1, \\ &\lim_{t \to a} \{ {}^{RL}D_{at}^{\alpha-2} f(t) \} = b_2, \\ &\qquad \dots \\ &\lim_{t \to a} \{ {}^{RL}D_{at}^{\alpha-n} f(t) \} = b_n. \end{aligned} $$

In spite of the fact that initial value problems with such initial conditions can be successfully solved mathematically, their solutions are practically useless, because there is no known physical interpretation for such types of initial conditions. An alternative solution to this conflict was proposed by M. Caputo [8]:

Definition 5. *Let* $a,b \in \mathbb{R}$, $a < b$, $n-1 < \alpha \leq n \in \mathbb{N}$, $f \in AC^n[a,b]$. *The Caputo fractional derivative of order* α*, is defined by*

$$ {}^{C}D_{at}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-(\alpha+1)} f^{(n)}(s) ds. $$

To describe material heterogeneities and structures with different scales, which cannot be well described by classical local theories or by fractional models with singular kernel, in [9] Caputo and Fabrizio introduced a new fractional approach:

Definition 6. *Let* $a,b \in \mathbb{R}$, $a < b$, $0 < \alpha < 1$, $f \in AC^1[a,b]$. *The Caputo-Fabrizio fractional derivative of order* α*, is defined by*

$$ D_{at}^{\alpha} f(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f'(s) ds. $$

All above defined fractional approaches have been used in numerous fields of science such as anomalous diffusion [10-12], circuit theory [13-15], image processing [16-19], and many others [20-40].

1.1 Basic properties of Riemann-Liouville fractional integral

Let us recall the Riemann-Liouville fractional integral of order $\alpha > 0$, given by

$$I_{at}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{3}$$

For $\alpha = 0$, we define $I_{at}^{0} f(t) = f(t)$. This definition is motivated by the following reasoning: suppose that $f(t) \in C^1([a,b])$. Then, calculating the integral (3) by the method of integration by parts, we obtain

$$I_{at}^{\alpha} f(t) = \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} f(a) + \frac{1}{\Gamma(\alpha+1)} \int_a^t (t-\tau)^{\alpha} f'(\tau) d\tau.$$

Therefore,

$$\lim_{\alpha \to 0} I_{at}^{\alpha} f(t) = f(a) + \int_a^t f'(\tau) d\tau = f(t). \tag{4}$$

One of the elemental properties of the Riemann-Liouville fractional integral is its linearity:

$$I_{at}^{\alpha}\big(\lambda f(t) + \beta g(t)\big) = \lambda I_{at}^{\alpha}\big(f(t)\big) + \beta I_{at}^{\alpha}\big(g(t)\big). \tag{5}$$

For $f \in L^p(a,b)$, $1 \le p \le \infty$, $\alpha, \beta > 0$, its corresponding composition property reads

$$I_{at}^{\alpha}\big(I_{at}^{\beta} f(t)\big) = I_{at}^{\alpha+\beta} f(t). \tag{6}$$

As examples, we recall the application of Riemann-Liouville fractional integral to the following elementary functions:

$$I_{0t}^{\alpha}\big(t^{\beta}\big) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \tag{7}$$

$$I_{0t}^{\alpha}(1) = \frac{1}{\Gamma(\alpha+1)} t^{\alpha}, \tag{8}$$

$$I_{0t}^{\alpha}\big(e^{\lambda t}\big) = t^{\alpha} \cdot E_{1,\alpha+1}(\lambda t), \quad \lambda \in \mathbb{R}. \tag{9}$$

Now, using the linearity property (5), the formula (9) for the exponential function and the representation of the sine function

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C},$$

it can be shown that

$$I_{0t}^{\alpha}\big(\sin \lambda t\big) = I_{0t}^{\alpha}\Big(\frac{e^{i\lambda t} - e^{-i\lambda t}}{2i}\Big) = \frac{1}{2i} I_{0t}^{\alpha}\big(e^{i\lambda t}\big) - \frac{1}{2i} I_{0t}^{\alpha}\big(e^{-i\lambda t}\big) = -\frac{1}{2} \cdot i \cdot t^{\alpha}\big(E_{1,\alpha+1}(i\lambda t) - E_{1,\alpha+1}(-i\lambda t)\big). \tag{10}$$

In the same manner, by using the formula

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C},$$

we obtain the expression for Riemann-Liouville integral of the cosine function:

$$I_{at}^{\alpha}\big(\cos \lambda t\big) = I_{at}^{\alpha}\Big(\frac{e^{i\lambda t} + e^{-i\lambda t}}{2}\Big) = \frac{1}{2} I_{at}^{\alpha}\big(e^{i\lambda t}\big) + \frac{1}{2} I_{at}^{\alpha}\big(e^{-i\lambda t}\big) = \frac{1}{2} t^{\alpha}\big(E_{1,\alpha+1}(i\lambda t) + E_{1,\alpha+1}(-i\lambda t)\big). \tag{11}$$

1.2 New fractional derivatives

In the present work, we shall introduce new definitions of fractional derivatives to improve theoretically the Riemann-Liouville and Caputo fractional derivatives. These defintions are motivated by the following reasoning: differentiating the Riemann-Liouville derivative

$$ {}^{RL}D_{at}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t(t-s)^{n-(\alpha+1)}f(s)ds, $$

and integrating by parts the Caputo derivative

$$ {}^{C}D_{at}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{n-(\alpha+1)}f^{(n)}(s)ds, $$

we obtain

$$ \begin{aligned} {}^{RL}D_{at}^{\alpha}f(t)&=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n-1}}{dt^{n-1}}\left[\frac{d}{dt}\int_a^t(t-s)^{n-(\alpha+1)}f(s)ds\right]\\ &=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n-1}}{dt^{n-1}}\left[0^{n-(\alpha+1)}\cdot f(t)+[n-(\alpha+1)]\int_a^t(t-s)^{n-(\alpha+2)}f(s)ds\right], \end{aligned} \tag{12} $$

and

$$ \begin{aligned} {}^{C}D_{at}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}&\Bigg[0^{n-(\alpha+1)}\cdot f^{(n-1)}(t)-(t-a)^{n-(\alpha+1)}f^{(n-1)}(a)\\ &+[n-(\alpha+1)]\int_a^t(t-s)^{n-(\alpha+2)}f^{(n-1)}(s)ds\Bigg], \end{aligned} \tag{13} $$

respectively. Since $n-(\alpha+1)<0$, the terms $0^{n-(\alpha+1)}\cdot f(t)$ and $0^{n-(\alpha+1)}\cdot f^{(n-1)}(t)$ of (12) and (13), respectively, are not defined. To avoid this issue, we propose the following definitions:

Definition 7. *Let $a,b\in\mathbb{R}$, $a<b$, $n-1<\alpha\le n\in\mathbb{N}$, $f\in AC^n[a,b]$. The new fractional derivative in the Riemann-Liouville sense of order α, is defined by*

$$ {}^{ARL}D_{at}^{\alpha,n}f(t):=\frac{1}{\Gamma(n-\alpha)}\frac{d^{2n}}{dt^{2n}}\int_a^t(t-s)^{n+\alpha-1}f(s)ds. \tag{14} $$

Definition 8. *Let $a,b\in\mathbb{R}$, $a<b$, $n-1<\alpha\le n\in\mathbb{N}$, $f\in AC^n[a,b]$. The new fractional derivative in the Caputo sense of order. α, is defined by*

$$ {}^{AC}D_{at}^{\alpha,n}f(t):=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_a^t(t-s)^{n+\alpha-1}f^{(n)}(s)ds. \tag{15} $$

2 Main properties

2.1 Basic properties of new fractional order derivatives

One of the important properties which characterize the fractional derivatives is the linearity:

Theorem 1. *Let a, $\mu,\beta\in\mathbb{R}$, $n-1<\alpha\leq n\in\mathbb{N}$, $f,g\in AC^n[a,b]$. Then*

$$ {}^{ARL}D_{at}^{\alpha,n}(\mu f(t)+\beta g(t))=\mu\cdot{}^{ARL}D_{at}^{\alpha,n}f(t)+\beta\cdot{}^{ARL}D_{at}^{\alpha,n}g(t), \tag{16}$$

$$ {}^{AC}D_{at}^{\alpha,n}(\mu f(t)+\beta g(t))=\mu\cdot{}^{AC}D_{at}^{\alpha,n}f(t)+\beta\cdot{}^{AC}D_{at}^{\alpha,n}g(t). \tag{17}$$

Proof. From (14), we obtain

$$\begin{aligned} {}^{ARL}D_{at}^{\alpha,n}(\mu f(t)+\beta g(t)) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^{2n}}{dt^{2n}}\int_a^t(t-\tau)^{n+\alpha-1}(\mu f(\tau)+\beta g(\tau))d\tau \\ &= \frac{\mu}{\Gamma(n-\alpha)}\frac{d^{2n}}{dt^{2n}}\int_a^t(t-\tau)^{n+\alpha-1}f(\tau)d\tau+\frac{\beta}{\Gamma(n-\alpha)}\frac{d^{2n}}{dt^{2n}}\int_a^t(t-\tau)^{n+\alpha-1}g(\tau)d\tau \\ &= \mu\cdot{}^{ARL}D_{at}^{\alpha,n}f(t)+\beta\cdot{}^{ARL}D_{at}^{\alpha,n}g(t). \end{aligned}$$

Similar to the proof of the equality (16), we have (17).

In the two following theorems we give the expressions of the Laplace transform of the defined fractional derivatives:

Theorem 2. *Let $n-1<\alpha\leq n\in\mathbb{N}$. Then*

$$\mathfrak{L}\{{}^{ARL}D_{0t}^{\alpha,n}f(t)\}=s^{n-\alpha}\cdot\mathfrak{L}\{f\}-\frac{1}{\Gamma(n-\alpha)}\cdot\sum_{k=0}^{2n-1}s^{2n-1-k}\cdot\frac{d^k}{dt^k}\left\{\int_0^t(t-s)^{n+\alpha-1}f(s)ds\right\}_{t=0}.$$

Proof. Applying the Laplace transform to the formula (14) and taking into account the formula for the Laplace transform of the derivative of any integer order, we have

$$\begin{aligned} \mathfrak{L}\{{}^{ARL}D_{0t}^{\alpha,n}f(t)\} &= \frac{1}{\Gamma(n-\alpha)}\cdot\mathfrak{L}\left\{\frac{d^{2n}}{dt^{2n}}\int_0^t(t-s)^{n+\alpha-1}f(s)ds\right\} \\ &= \frac{1}{\Gamma(n-\alpha)}\cdot\left\{s^{2n}\cdot\mathfrak{L}\left\{\int_0^t(t-s)^{n+\alpha-1}f(s)ds\right\}-\sum_{k=0}^{2n-1}s^{2n-1-k}\cdot\frac{d^k}{dt^k}\left\{\int_0^t(t-s)^{n+\alpha-1}f(s)ds\right\}_{t=0}\right\}. \end{aligned} \tag{18}$$

Combining the property of the Laplace transform of the convolution with the second equality of (18), we obtain

$$\begin{aligned} \mathfrak{L}\{{}^{ARL}D_{0t}^{\alpha,n}f(t)\} &= \frac{1}{\Gamma(n-\alpha)}\cdot s^{2n}\cdot\mathfrak{L}\{s^{n+\alpha-1}\}\cdot\mathfrak{L}\{f(t)\} \\ &\quad -\frac{1}{\Gamma(n-\alpha)}\sum_{k=0}^{2n-1}s^{2n-1-k}\cdot\frac{d^k}{dt^k}\left\{\int_0^t(t-s)^{n+\alpha-1}f(s)ds\right\}_{t=0}. \end{aligned}$$

Nowing that $\mathfrak{L}\{t^{n+\alpha-1}\}(s)=s^{-n-\alpha}\Gamma(n-\alpha)$, we get

$$\mathfrak{L}\{{}^{ARL}D_{0t}^{\alpha,n}f(t)\}=s^{n-\alpha}\cdot\mathfrak{L}\{f(t)\}-\frac{1}{\Gamma(n-\alpha)}\sum_{k=0}^{2n-1}s^{2n-1-k}\cdot\frac{d^k}{dt^k}\left\{\int_0^t(t-s)^{n+\alpha-1}f(s)ds\right\}_{t=0}.$$

Theorem 3. *Let $n-1<\alpha\leq n\in\mathbb{N}$. Then*

$$\mathfrak{L}\{{}^{AC}D_{0t}^{\alpha,n}f(t)\}=\frac{\Gamma(\alpha)}{\Gamma(n-\alpha)}\cdot\left[\prod_{k=1}^{n}(n+\alpha-k)\right]\cdot s^{-\alpha}\cdot\left\{s^n\mathfrak{L}\{f(t)\}-\sum_{k=0}^{n-1}s^{n-1-k}f^{(k)}(0)\right\}.$$

Proof. Performing repeatedly the differentiation n times, we obtain

$$\frac{d^n}{dt^n}\int_a^t(t-s)^{n+\alpha-1}f^{(n)}(s)ds=\left[\prod_{k=1}^{n}(n+\alpha-k)\right]\cdot\int_a^t(t-s)^{\alpha-1}f^{(n)}(s)ds. \tag{19}$$

Applying $\dfrac{1}{\Gamma(n-\alpha)}$ into the both sides of (19), yields

$$ {}^{AC}D_{0t}^{\alpha,n} f(t) = \frac{1}{\Gamma(n-\alpha)} \left[\prod_{k=1}^{n} (n+\alpha-k) \right] \cdot \int_a^t (t-s)^{\alpha-1} f^{(n)}(s) ds. \tag{20} $$

Now, applying the Laplace transform to the equality (20), we obtain

$$ \mathfrak{L}\{ {}^{AC}D_{0t}^{\alpha,n} f(t)\} = \frac{1}{\Gamma(n-\alpha)} \cdot \left[\prod_{k=1}^{n} (n+\alpha-k) \right] \cdot \mathfrak{L}\left\{ \int_a^t (t-s)^{\alpha-1} f^{(n)}(s) ds \right\}. \tag{21} $$

Combining the property of the Laplace transform of the convolution with (21), we get

$$ \mathfrak{L}\{ {}^{AC}D_{0t}^{\alpha,n} f(t)\} = \frac{1}{\Gamma(n-\alpha)} \cdot \left[\prod_{k=1}^{n} (n+\alpha-k) \right] \cdot \mathfrak{L}\{t^{\alpha-1}\} \cdot \mathfrak{L}\{f^{(n)}(t)\}. $$

Nowing that $\mathfrak{L}\{t^{\alpha-1}\}(s) = s^{-\alpha}\Gamma(\alpha)$ and using the Laplace transform of the derivative of any integer order, we obtain

$$ \mathfrak{L}\{ {}^{AC}D_{0t}^{\alpha,n} f(t)\} = \frac{1}{\Gamma(n-\alpha)} \cdot \left[\prod_{k=1}^{n} (n+\alpha-k) \right] \cdot s^{-\alpha}\Gamma(\alpha) \left\{ s^n \mathfrak{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0) \right\}, $$

as required.

To prove that the proposed fractional derivatives coincide with the derivative of integer order $\frac{d^n}{dt^n}$, for $\alpha = n \in \mathbb{N}$, we need the two following results.

Lemma 1. *Let $a \in \mathbb{R}$, $n-1 < \alpha \le n \in \mathbb{N}$. Then, the operator $d_{at}^{\alpha} f(t)$ defined by*

$$ d_{at}^{\alpha} f(t) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n+\alpha-1} f(s) ds, $$

satisfies

$$ d_{at}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \cdot \left[\prod_{i=0}^{n-1} (i+\alpha) \right] \cdot \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{22} $$

Proof. The equality (22) will be proved by using the method of mathematical induction. For $n = 1$, we have

$$ d_{at}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{\alpha} f(s) ds = \frac{1}{\Gamma(1-\alpha)} \cdot \alpha \int_a^t (t-s)^{\alpha-1} f(s) ds = \frac{1}{\Gamma(1-\alpha)} \left[\prod_{i=0}^{1-1} (i+\alpha) \right] \int_a^t (t-s)^{\alpha-1} f(s) ds. $$

Thus, (22) is true for $n = 1$. Let suppose that (22) is true for $n \in \mathbb{N}$ $(n > 1)$. To conclude with the proof we have to demonstrate that (22) is also true for $n+1$. We have

$$\begin{aligned} d_{at}^{\alpha} f(t) &= \frac{1}{\Gamma(n+1-\alpha)} \frac{d^{n+1}}{dt^{n+1}} \int_a^t (t-s)^{n+1-(1-\alpha)} f(s) ds \\ &= \frac{1}{\Gamma(n+1-\alpha)} \frac{d^n}{dt^n} \left[\frac{d}{dt} \int_a^t (t-s)^{n+\alpha} f(s) ds \right] \\ &= \frac{n+\alpha}{\Gamma(n+1-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-(1-\alpha)} f(s) ds \\ &= \frac{n+\alpha}{\Gamma(n+1-\alpha)} \cdot \left[\prod_{i=0}^{n-1} (i+\alpha) \right] \cdot \int_a^t (t-s)^{\alpha-1} f(s) ds \\ &= \frac{1}{\Gamma(n+1-\alpha)} \cdot \left[\prod_{i=0}^{n} (i+\alpha) \right] \cdot \int_a^t (t-s)^{\alpha-1} f(s) ds. \end{aligned}$$

Lemma 2. *Let $a \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then*

$$\lim_{\alpha \to n} d_{at}^{\alpha} f(t) = f(t). \tag{23}$$

Proof. By rewriting formula (22) as

$$\begin{aligned} d_{at}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \cdot \left[\prod_{i=0}^{n-1} (i+\alpha) \right] \cdot \int_a^t (t-s)^{\alpha-1} f(s) ds \\ &= \frac{\Gamma(\alpha) \prod_{i=0}^{n-1} (i+\alpha)}{\Gamma(n-\alpha)} \cdot \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds \\ &= \frac{\Gamma(n+\alpha)}{\Gamma(n-\alpha)} \cdot \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \end{aligned}$$

and using the fact that $\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds \to f(t)$ for $\alpha \to 0$ (see (4)), we obtain (23).

Now, we are in conditions to prove that ${}^{ARL}D_{at}^{\alpha,n} f(t) \to f^{(n)}(t)$ and ${}^{AC}D_{at}^{\alpha,n} f(t) \to f^{(n)}(t)$, for $\alpha \to n$:

Theorem 4. *Let $a \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then*

$$\begin{aligned} \lim_{\alpha \to n} \{ {}^{ARL}D_{at}^{\alpha,n} f(t) \} &= f^{(n)}(t), \\ \lim_{\alpha \to n} \{ {}^{AC}D_{at}^{\alpha,n} f(t) \} &= f^{(n)}(t). \end{aligned}$$

Proof. Formulas (14) and (15) can be rewritten as

$${}^{ARL}D_{at}^{\alpha,n} f(t) = \frac{d^n}{dt^n} \left(d_{at}^{\alpha} f(t) \right), \tag{24}$$

and

$${}^{AC}D_{at}^{\alpha,n} f(t) = d_{at}^{\alpha} \left(f^{(n)}(t) \right), \tag{25}$$

respectively. Taking limit for $\alpha \to n$, we obtain from (24)-(25) that

$$\begin{aligned} \lim_{\alpha \to n} \left\{ {}^{ARL}D_{at}^{\alpha,n} f(t) \right\} &= \lim_{\alpha \to n} \left\{ \frac{d^n}{dt^n} \left(d_{at}^{\alpha} f(t) \right) \right\} = \frac{d^n}{dt^n} \left\{ \lim_{\alpha \to n} \left(d_{at}^{\alpha} f(t) \right) \right\} = f^{(n)}(t), \\ \lim_{\alpha \to n} \{ {}^{AC}D_{at}^{\alpha,n} f(t) \} &= \lim_{\alpha \to n} \{ d_{at}^{\alpha} \left(f^{(n)}(t) \right) \} = f^{(n)}(t), \end{aligned}$$

as required.

To give the relation between the two proposed operators, we need the following results:

Lemma 3. *Let $a \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then,*

$$ {}^{AC}D_{at}^{\alpha,n} f(t) = -\frac{1}{\Gamma(n-\alpha)} \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+k+1-n)} (t-a)^{\alpha+k-n} f^{(k)}(a) + \frac{1}{\Gamma(n-\alpha)} \left[\prod_{k=0}^{n-1} (\alpha+k)\right] \cdot \frac{d^n}{dt^n} \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{26} $$

Proof. Performing repeatedly the method of integration by parts n times, we obtain

$$ \int_a^t (t-s)^{n+\alpha-1} f^{(n)}(s) ds = -\sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+k+1)} (t-a)^{\alpha+k} f^{(k)}(a) + \left[\prod_{k=0}^{n-1} (\alpha+k)\right] \cdot \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{27} $$

Applying the operator $\frac{d^n}{dt^n}$ to the both sides of (27) and then multiplying by $\frac{1}{\Gamma(n-\alpha)}$, we get (26).

Lemma 4. *Let $a \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then,*

$$ {}^{ARL}D_{at}^{\alpha,n} f(t) = \frac{1}{\Gamma(n-\alpha)} \left[\prod_{k=1}^{n} (n+\alpha-k)\right] \cdot \frac{d^n}{dt^n} \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{28} $$

Proof. Performing repeatedly the differentiation n times, we obtain

$$ \frac{d^n}{dt^n} \int_a^t (t-s)^{n+\alpha-1} f(s) ds = \left[\prod_{k=1}^{n} (n+\alpha-k)\right] \cdot \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{29} $$

Applying $\frac{d^n}{dt^n}$ into the both sides of (29) and then multiplying the obtained result by $\frac{1}{\Gamma(n-\alpha)}$, we get (28).

Now we are in conditions to establish the relation between the two proposed operators.

Theorem 5. *Let $a \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then,*

$$ {}^{ARL}D_{at}^{\alpha,n} f(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+k+1-n)} (t-a)^{\alpha+k-n} f^{(k)}(a) + {}^{AC}D_{at}^{\alpha,n} f(t). \tag{30} $$

Proof. Considering that $\prod_{k=0}^{n-1}(\alpha+k) = \prod_{k=1}^{n}(n+\alpha-k)$ and combining (26) with (28), we obtain (30).

Next, we formulate results to establish the composition of the proposed operators with derivative of integer order.

Theorem 6. *Let $a \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then,*

$$ \frac{d^n}{dt^n} \{ {}^{AC}D_{at}^{\alpha,n} f(t) \} = {}^{ARL}D_{at}^{\alpha,n} \left(f^{(n)}(t)\right). \tag{31} $$

Proof. We have

$$ \frac{d^n}{dt^n} \{ {}^{AC}D_{at}^{\alpha,n} f(t) \} = \frac{d^n}{dt^n} \left\{ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n+\alpha-1} f^{(n)}(s) ds \right\} = \frac{1}{\Gamma(n-\alpha)} \frac{d^{2n}}{dt^{2n}} \int_a^t (t-s)^{n+\alpha-1} f^{(n)}(s) ds. \tag{32} $$

Formula (31) follows from (32).

Theorem 7. *Let $a \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then*

$$\frac{d^n}{dt^n}\{^{ARL}D_{at}^{\alpha,n} f(t)\} = \frac{\prod_{k=1}^{n}(n+\alpha-k)}{\Gamma(n-\alpha)} \cdot \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+k+1)(t-a)^{\alpha+k-2n}}{\Gamma(\alpha+k+1-2n)\prod_{i=0}^{k}(\alpha+i)} f^{(k)}(a) + {}^{ARL}D_{at}^{\alpha,n} f^{(n)}(t). \tag{33}$$

Proof. From (14), we obtain

$$\begin{aligned}\frac{d^n}{dt^n}\{^{ARL}D_{at}^{\alpha,n} f(t)\} &= \frac{d^n}{dt^n}\left\{\frac{1}{\Gamma(n-\alpha)}\frac{d^{2n}}{dt^{2n}}\int_a^t (t-s)^{n+\alpha-1} f(s)ds\right\} \\ &= \frac{d^{2n}}{dt^{2n}}\left\{\frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_a^t (t-s)^{n+\alpha-1} f(s)ds\right\}.\end{aligned} \tag{34}$$

Combining (29) with the second equality of (34), we get

$$\frac{d^n}{dt^n}\{^{ARL}D_{at}^{\alpha,n} f(t)\} = \frac{1}{\Gamma(n-\alpha)}\left[\prod_{k=1}^{n}(n+\alpha-k)\right] \cdot \frac{d^{2n}}{dt^{2n}}\int_a^t (t-s)^{\alpha-1} f(s)ds. \tag{35}$$

Performing repeatedly the method of integration by parts n times, we obtain

$$\int_a^t (t-s)^{\alpha-1} f(s)ds = \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha+k}}{\prod_{i=0}^{k}(\alpha+i)} f^{(k)}(a) + \frac{1}{\prod_{k=0}^{n-1}(\alpha+k)}\int_a^t (t-s)^{n+\alpha-1} f^{(n)}(s)ds. \tag{36}$$

From (35) and (36), we obtain

$$\begin{aligned}\frac{d^n}{dt^n}\{^{ARL}D_{at}^{\alpha,n} f(t)\} &= \frac{1}{\Gamma(n-\alpha)}\left[\prod_{k=1}^{n}(n+\alpha-k)\right] \cdot \frac{d^n}{dt^n}\left\{\frac{d^n}{dt^n}\left[\sum_{k=0}^{n-1}\frac{(t-a)^{\alpha+k}}{\prod_{i=0}^{k}(\alpha+i)} f^{(k)}(a)\right.\right. \\ &\quad \left.\left.+ \frac{1}{\prod_{k=0}^{n-1}(\alpha+k)}\int_a^t (t-s)^{n+\alpha-1} f^{(n)}(s)ds\right]\right\} \\ &= \left[\prod_{k=1}^{n}(n+\alpha-k)\right] \cdot \frac{d^n}{dt^n}\left\{\frac{1}{\Gamma(n-\alpha)}\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+k+1)(t-a)^{\alpha+k-n}}{\Gamma(\alpha+k+1-n)\prod_{i=0}^{k}(\alpha+i)} f^{(k)}(a)\right. \\ &\quad \left.+ \frac{1}{\prod_{k=0}^{n-1}(\alpha+k)}\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n+\alpha-1} f^{(n)}(s)ds\right\}.\end{aligned} \tag{37}$$

Combining (15) with (37), it follows

$$\frac{d^n}{dt^n}\{{}^{ARL}D_{at}^{\alpha,n}f(t)\} = \left[\prod_{k=1}^{n}(n+\alpha-k)\right]\cdot\frac{d^n}{dt^n}\left\{\frac{1}{\Gamma(n-\alpha)}\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+k+1)(t-a)^{\alpha+k-n}}{\Gamma(\alpha+k+1-n)\prod_{i=0}^{k}(\alpha+i)}f^{(k)}(a)\right.$$
$$\left.+\frac{1}{\prod_{k=0}^{n-1}(\alpha+k)}\{{}^{AC}D_{at}^{\alpha,n}f(t)\}\right\}$$
$$= \left[\prod_{k=1}^{n}(n+\alpha-k)\right]\left\{\frac{1}{\Gamma(n-\alpha)}\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+k+1)(t-a)^{\alpha+k-2n}}{\Gamma(\alpha+k+1-2n)\prod_{i=0}^{k}(\alpha+i)}f^{(k)}(a)\right.$$
$$\left.+\frac{1}{\prod_{k=0}^{n-1}(\alpha+k)}\frac{d^n}{dt^n}\{{}^{AC}D_{at}^{\alpha,n}f(t)\}\right\}. \tag{38}$$

From (31) and (38), we obtain

$$\frac{d^n}{dt^n}\{{}^{ARL}D_{at}^{\alpha,n}f(t)\} = \frac{\prod_{k=1}^{n}(n+\alpha-k)}{\Gamma(n-\alpha)}\cdot\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+k+1)(t-a)^{\alpha+k-2n}}{\Gamma(\alpha+k+1-2n)\prod_{i=0}^{k}(\alpha+i)}f^{(k)}(a)$$
$$+\frac{\prod_{k=1}^{n}(n+\alpha-k)}{\prod_{k=0}^{n-1}(\alpha+k)}\cdot\{{}^{ARL}D_{at}^{\alpha,n}f^{(n)}(t)\}. \tag{39}$$

Since $\prod_{k=1}^{n}(n+\alpha-k) = \prod_{k=0}^{n-1}(\alpha+k)$, then the equality (33) follows from (39).

Theorem 8. *Let $a\in\mathbb{R}$, $n-1<\alpha\le n\in\mathbb{N}$, $f^{(k)}(a)=0$ for $k=0,1,\cdots,n-1$. Then*

$$\frac{d^n}{dt^n}\{{}^{ARL}D_{at}^{\alpha,n}f(t)\} = {}^{ARL}D_{at}^{\alpha,n}\left(\frac{d^n}{dt^n}f(t)\right). \tag{40}$$

*Proof.*Equality (40) follows from (33).

Theorem 9. *Let $a\in\mathbb{R}$, $n-1<\alpha\le n\in\mathbb{N}$. Then,*

$${}^{ARL}D_{at}^{\alpha,n}(f^{(n)}(t)) = \frac{1}{\Gamma(n-\alpha)}\left[\prod_{k=1}^{n}(n+\alpha-k)\right]\frac{d^n}{dt^n}\int_a^t(t-s)^{\alpha-1}f^{(n)}(s)ds. \tag{41}$$

Proof. Using the formula (19), we obtain

$${}^{ARL}D_{at}^{\alpha,n}(f^{(n)}(t)) = \frac{1}{\Gamma(n-\alpha)}\frac{d^{2n}}{dt^{2n}}\int_a^t(t-s)^{n+\alpha-1}f^{(n)}(s)ds$$
$$= \frac{d^n}{dt^n}\left\{\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t(t-s)^{n+\alpha-1}f^{(n)}(s)ds\right\}$$
$$= \frac{1}{\Gamma(n-\alpha)}\left[\prod_{k=1}^{n}(n+\alpha-k)\right]\frac{d^n}{dt^n}\int_a^t(t-s)^{\alpha-1}f^{(n)}(s)ds.$$

Theorem 10. *Let $a \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then,*

$$ {}^{AC}D_{at}^{\alpha,n}\left(f^{(n)}(t)\right) = -\frac{1}{\Gamma(n-\alpha)} \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha-k)} (t-a)^{\alpha-k-1} f^{(2n-1-k)}(a) + \frac{d^n}{dt^n}\{{}^{AC}D_{at}^{\alpha,n} f(t)\}. \tag{42} $$

*Proof.*From formula (15), we obtain

$$ {}^{AC}D_{at}^{\alpha,n}\left(f^{(n)}(t)\right) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n+\alpha-1} f^{(2n)}(s) ds. \tag{43} $$

Calculating the integral

$$ \int_a^t (t-s)^{n+\alpha-1} f^{(2n)}(s) ds, \tag{44} $$

n times by parts, we have

$$ \begin{aligned} \int_a^t (t-s)^{n+\alpha-1} f^{(2n)}(s) ds = &-\sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n-k)} (t-a)^{\alpha+n-1-k} f^{(2n-1-k)}(a) \\ &+ \left[\prod_{k=0}^{n-1} (\alpha+n-k)\right] \cdot \int_a^t (t-s)^{\alpha-1} f^{(n)}(s) ds. \end{aligned} \tag{45} $$

Combining (43) with (45), we have

$$ \begin{aligned} {}^{AC}D_{at}^{\alpha,n}\left(f^{(n)}(t)\right) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n+\alpha-1} f^{(2n)}(s) ds \\ &= -\frac{1}{\Gamma(n-\alpha)} \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha-k)} (t-a)^{\alpha-k-1} f^{(2n-1-k)}(a) \\ &\quad + \frac{1}{\Gamma(n-\alpha)} \left[\prod_{k=0}^{n-1} (\alpha+n-k)\right] \cdot \frac{d^n}{dt^n} \int_a^t (t-s)^{\alpha-1} f^{(n)}(s) ds. \end{aligned} \tag{46} $$

From (41) and (46), we have

$$ {}^{AC}D_{at}^{\alpha,n}\left(f^{(n)}(t)\right) = -\frac{1}{\Gamma(n-\alpha)} \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha-k)} (t-a)^{\alpha-k-1} f^{(2n-1-k)}(a) + {}^{ARL}D_{at}^{\alpha}\left(f^{(n)}(t)\right). \tag{47} $$

Using (31), we get

$$ {}^{AC}D_{at}^{\alpha,n}\left(f^{(n)}(t)\right) = -\frac{1}{\Gamma(n-\alpha)} \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha-k)} (t-a)^{\alpha-k-1} f^{(2n-1-k)}(a) + \frac{d^n}{dt^n}\{{}^{AC}D_{at}^{\alpha,n} f(t)\}. $$

Theorem 11. *Let $a \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$, $f^{(2n-1-k)}(a) = 0$ for $k = 0, 1, \cdots, n-1$. Then,*

$$ \frac{d^n}{dt^n}\{{}^{AC}D_{at}^{\alpha,n} f(t)\} = {}^{AC}D_{at}^{\alpha,n}\left(\frac{d^n}{dt^n} f(t)\right). \tag{48} $$

Proof. Formula (48) follows from (42).

Next, some examples of the new fractional derivative in the Riemann-Liouville sense are discussed, that is, the constant, the power and the exponential function, as well as the sine and cosine function.

Theorem 12. *Let $\alpha \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then,*

$$ {}^{ARL}D_{0t}^{\alpha,n} 1 = \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)} \left[\prod_{k=1}^{n} (n+\alpha-k) \right] \frac{1}{\Gamma(\alpha+1-n)} t^{\alpha-n}. \tag{49} $$

Proof. On the one hand, from the formula (8), we have

$$ \frac{d^n}{dt^n} \left(I_{0t}^{\alpha} 1 \right) = \frac{1}{\Gamma(\alpha+1-n)} t^{\alpha-n}. \tag{50} $$

On the other hand, combining the relation (28) with (50), we obtain (49).

For the potential function, we have the following result:

Theorem 13. *Let $\beta \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then,*

$$ {}^{ARL}D_{0t}^{\alpha,n} t^{\beta} = \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)} \left[\prod_{k=1}^{n} (n+\alpha-k) \right] \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1-n)} t^{\beta+\alpha-n}. \tag{51} $$

Proof. On the one hand, from the relation (28), we have

$$ \begin{aligned} {}^{ARL}D_{0t}^{\alpha,n} t^{\beta} &= \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)} \left[\prod_{k=1}^{n} (n+\alpha-k) \right] \cdot \frac{1}{\Gamma(\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{\alpha-1} s^{\beta} ds \\ &= \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)} \left[\prod_{k=1}^{n} (n+\alpha-k) \right] \cdot \frac{d^n}{dt^n} \left(I_{0t}^{\alpha} t^{\beta} \right). \end{aligned} \tag{52} $$

On the other hand, from (7), we get

$$ \frac{d^n}{dt^n} \left(I_{0t}^{\alpha} t^{\beta} \right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1-n)} t^{\beta+\alpha-n}. \tag{53} $$

Combining (52) with (53), we obtain (51).

For the exponential function, we formulate the following theorem:

Theorem 14. *Let $\lambda \in \mathbb{R}$, $n-1 < \alpha \leq n \in \mathbb{N}$. Then,*

$$ \begin{aligned} {}^{ARL}D_{0t}^{\alpha,n} \left(e^{\lambda t} \right) &= \frac{\Gamma(\alpha) \cdot t^{\alpha-n}}{\Gamma(n-\alpha)} \left[\prod_{k=1}^{n} (\alpha+n-k) \right] \cdot \\ &\cdot \left\{ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} E_{1,\alpha+1}(\lambda t) + \sum_{k=0}^{\infty} \frac{k! (\lambda t)^k}{\Gamma(k+\alpha+1)\Gamma(k-n+1)} \right\}. \end{aligned} \tag{54} $$

Proof. On the one hand, from the relation (28), we obtain

$$ \begin{aligned} {}^{ARL}D_{0t}^{\alpha,n} \left(e^{\lambda t} \right) &= \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)} \left[\prod_{k=1}^{n} (n+\alpha-k) \right] \cdot \frac{1}{\Gamma(\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{\alpha-1} e^{\lambda s} ds \\ &= \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)} \left[\prod_{k=1}^{n} (n+\alpha-k) \right] \cdot \frac{d^n}{dt^n} \left(I_{0t}^{\alpha} (e^{\lambda s}) \right). \end{aligned} \tag{55} $$

On the other hand, from (9), we have

$$ \frac{d^n}{dt^n} \left(I_{0t}^{\alpha} (e^{\lambda s}) \right) = t^{\alpha-n} \left[\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} E_{1,\alpha+1}(\lambda t) + \sum_{k=0}^{\infty} \frac{k! (\lambda t)^k}{\Gamma(k+\alpha+1)\Gamma(k-n+1)} \right]. \tag{56} $$

Equality (54) follows from the combination of (55) with (56).

The expressions of the new fractional derivative in Riemann-Liouville sense for sine and cosine function are given by the two following theorems:

Theorem 15. *Let $\lambda \in \mathbb{R}$, $n-1<\alpha\leq n\in\mathbb{N}$. Then,*

$$
\begin{aligned}
{}^{ARL}D_{0t}^{\alpha,n}(\sin\lambda t) = &-\frac{i\cdot\Gamma(\alpha)\cdot t^{\alpha-n}}{2\cdot\Gamma(n-\alpha)}\left[\prod_{k=1}^{n}(\alpha+n-k)\right]\cdot\left\{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\left[E_{1,\alpha+1}(i\lambda t)-E_{1,\alpha+1}(-i\lambda t)\right]\right.\\
&\left.+\sum_{k=0}^{\infty}\frac{k!(i\lambda t)^k}{\Gamma(k+\alpha+1)\Gamma(k-n+1)}-\sum_{k=0}^{\infty}\frac{k!(-i\lambda t)^k}{\Gamma(k+\alpha+1)\Gamma(k-n+1)}\right\}.
\end{aligned}
\tag{57}
$$

Proof. On the one hand, from the relation (28), we obtain

$$
\begin{aligned}
{}^{ARL}D_{0t}^{\alpha,n}(\sin\lambda t) &= \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)}\left[\prod_{k=1}^{n}(n+\alpha-k)\right]\cdot\frac{1}{\Gamma(\alpha)}\frac{d^n}{dt^n}\int_0^t(t-s)^{\alpha-1}\sin\lambda s ds\\
&= \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)}\left[\prod_{k=1}^{n}(n+\alpha-k)\right]\cdot\frac{d^n}{dt^n}\left(I_{0t}^{\alpha}(\sin\lambda t)\right).
\end{aligned}
\tag{58}
$$

On the other hand from (10), we have

$$
\begin{aligned}
\frac{d^n}{dt^n}\left(I_{0t}^{\alpha}(\sin\lambda t)\right) = &-\frac{i}{2}\cdot t^{\alpha-n}\left\{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\left[E_{1,\alpha+1}(i\lambda t)-E_{1,\alpha+1}(-i\lambda t)\right]\right.\\
&\left.+\sum_{k=0}^{\infty}\frac{k!(i\lambda t)^k}{\Gamma(k+\alpha+1)\Gamma(k-n+1)}-\sum_{k=0}^{\infty}\frac{k!(-i\lambda t)^k}{\Gamma(k+\alpha+1)\Gamma(k-n+1)}\right\}.
\end{aligned}
\tag{59}
$$

Combining (58) with (59), we obtain (57).

Theorem 16. *Let $\lambda \in \mathbb{R}$, $n-1<\alpha\leq n\in\mathbb{N}$. Then,*

$$
\begin{aligned}
{}^{ARL}D_{0t}^{\alpha,n}(\cos\lambda t) = &\frac{\Gamma(\alpha)\cdot t^{\alpha-n}}{2\cdot\Gamma(n-\alpha)}\left[\prod_{k=1}^{n}(\alpha+n-k)\right]\cdot\left\{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\left[E_{1,\alpha+1}(i\lambda t)+E_{1,\alpha+1}(-i\lambda t)\right]\right.\\
&\left.+\sum_{k=0}^{\infty}\frac{k!(i\lambda t)^k}{\Gamma(k+\alpha+1)\Gamma(k-n+1)}+\sum_{k=0}^{\infty}\frac{k!(-i\lambda t)^k}{\Gamma(k+\alpha+1)\Gamma(k-n+1)}\right\}.
\end{aligned}
\tag{60}
$$

Proof. On the one hand, from the relation (28), we have

$$
\begin{aligned}
{}^{ARL}D_{0t}^{\alpha,n}(\cos\lambda t) &= \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)}\left[\prod_{k=1}^{n}(n+\alpha-k)\right]\cdot\frac{1}{\Gamma(\alpha)}\frac{d^n}{dt^n}\int_0^t(t-s)^{\alpha-1}\cos\lambda s ds\\
&= \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)}\left[\prod_{k=1}^{n}(n+\alpha-k)\right]\cdot\frac{d^n}{dt^n}\left(I_{0t}^{\alpha}(\cos\lambda t)\right).
\end{aligned}
\tag{61}
$$

On the other hand, from (11), we get

$$
\begin{aligned}
\frac{d^n}{dt^n}\left(I_{0t}^{\alpha}(\cos\lambda t)\right) = &\frac{1}{2}\cdot t^{\alpha-n}\left\{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\left[E_{1,\alpha+1}(i\lambda t)+E_{1,\alpha+1}(-i\lambda t)\right]\right.\\
&\left.+\sum_{k=0}^{\infty}\frac{k!(i\lambda t)^k}{\Gamma(k+\alpha+1)\Gamma(k-n+1)}+\sum_{k=0}^{\infty}\frac{k!(-i\lambda t)^k}{\Gamma(k+\alpha+1)\Gamma(k-n+1)}\right\}.
\end{aligned}
\tag{62}
$$

Combining (61) with (62), we obtain (60).

3 Conclusion

The aim of this paper was to suggest new fractional derivatives to improve theoretically the Riemann-Liouville and Caputo fractional derivatives. In this sense, one of the derivatives is based upon the Riemann-Liouville viewpoint and the other one on the Caputo approach. Also some properties have been given to illustrate results.

Conflict of Interest. The author declares that he has no conflict of interest.

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