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Further properties of L-closed topological spaces

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Abstract: The main purpose of this research paper is to discuss properties of L-closed spaces in topology and propose further properties concerning this concept. In addition, we explored the relationships between L-closed spaces and other topological spaces.

Keywords: L-closed space, L-closed neighborhood, L-closed property.

1 Introduction

In 1983, Hdeib and Pareek [6] introduced a new topological space called L-closed space in which each Lindelöf subset of a topological space (X,τ) is closed. Lots of results concerning the new notion are obtained and they posed two queries, the first query is: "Does there exists a regular L-closed space which are not a P-spaces?"

The second one is: "In an L-closed space, every countable subset is closed, when is the inverse true?". In 1985, two mathematicians Henriksen and Woods [17] presented their answers to the queries within the Tychonoff spaces.

In this paper, we will denote a topological space (X,τ) by X, $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real and natural numbers respectively. Moreover, τ_{coc}, τ_{dis}, τ_u, τ_s, τ_{cof} , τ_l and τ_r, will denote cocountable, discrete, usual, Sorgenfrey, cofinite, left ray, and right ray opologies. Finally, the closure and interior of a set in a space say U will be denoted by clU and $intU$ respectively.

2 Preliminaries

Studying compact and Lindelöf topological spaces requires a deep understand of the nature of their covers, Fletcher and other topologists [5] displayed the definitions of open cover where a cover (indexed family) $C = \{U_\alpha : \alpha \in \Lambda\}$ of a non-empty space X is a collection of subsets whose union is X.

Lots of topologists focused on the importance of covers in determining the nature and the type of a space, for instance, Fletcher [5], Hoyle, Patty and Kim [9] proved that the product of pairwise compact spaces is not a pairwise compact space.

Lindelöf spaces, introduced by Alexandroff and Urysohn in 1929, have a highly expressive role in topology, it generalizes compact spaces.

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The most familiar concepts in topology are ”Separation axioms” that define topological classes and distinguish disjoint sets and distinct points [2]. Such classes of spaces are given the names T_1, T_2, T_3, and T_4 as an abbreviation of a German word ”Trennung” which means separation. After there, system of numbering was extended to include T_0, $T_{2Â½}$, $T_{3Â½}$ (or T_π), T_5, and T_6 [3].

3 L-closed Spaces

In a space X, if every countable subset is closed, consequently every countable subset of such space is discrete, and every compact subset is finite [6], consequently every subset of L-closed space X is closed and every subspace of an L-closed space is also L-closed.

Each Housdorff space is L-closed space and if $\forall x$ in a space X has an L-closed neighborhood say W where $\exists$ W a Lindelöf subset of X containing x. Now, a regular space X. In regular spaces in which $\forall x$ has an L-closed neighborhood, a space will be L-closed. Moreover, a regular L-closed space is normal [14]. Now, a regular space X that can be represented as a countable union of subspaces and each of such subspaces has L-closeness property has itself L-closeness property.
In a hereditarily Lindelöf space X, X is L-closed if and only if it is countable and discrete. The sum $\bigoplus_{\alpha\in\Lambda} X_\alpha$ where $X_\alpha \neq \phi$ for some $\alpha\in\Lambda$ has L-closeness property if and only if each space X_α has the L-closeness property and Λ is a countable set. Furthermore, a subset $A\subset \bigoplus_{\alpha\in\Lambda} X_\alpha$X is L-closed $\Longleftrightarrow$ $A\cap X_\alpha$ is L-closed in X_α, $\alpha\in\Lambda$.

A space X is L_4-space [8] if $\forall A$ a Lindelöf subset of X, $\exists F$ a Lindelöf F_σ-subset of $X \ni A\subseteq F\subseteq \overline{A}$. Typically, each L-closed space is L_4-space and as a consequence we have that in each L-closed space, every Lindelöf F_σ-subset is closed. Now, if every Lindelöf subset of X is a F_σ-subset, then X is T_1 and each closed countable subset of X will be discrete.

A space X is L_1 [8] if every Lindelöf F_σ-subset of X is closed. Depending on this definition, an L-closed space is L_1space. The property of being pairwise L_1-space is hereditarily property with respect to F_σ-subsets.

A space X is L_3 [8] if every Lindelöf subset of X is an F_σ-set. From this definition, we conclude that the property of being a L_3-space is a hereditarily property and that X is L-closed $\Longleftrightarrow$ it is L_1 and L_3.

4 Product properties of L-closed spaces

Proposition 1. *Let (X,τ) and (Y,σ) be two spaces such that Y is an L-closed space. If f:$(X,\tau)\rightarrow(Y,\sigma)$ is a continuous onto function, then (X,τ) is an L-closed space.*

Proof. If f:$(X,\tau)\rightarrow(Y,\sigma)$ is a continuous onto function. Let (Y,σ) be an L-closed space, let F be a Lindelöf subset of X, then f(F) is Lindelöf since f is a continuous function.

But (Y,σ) is an L-closed space, thus f(F) is a closed subset of Y, and F=f $^{-1}$(f(F)) is a closed subset of X because f is a one to one function. Therefore, (X,τ) is an L-closed space. Now, any L-closed topological space is homeomorphic to itself.

Proposition 2. *Let (X,τ) and (Y,σ) be two spaces such that (X,τ) is Lindelöf, (Y,σ) is L-closed, if f: $(X,\tau)\rightarrow$ (Y,σ) is an onto continuous function, then f is homeomorphism.*

Proof. We claim that f is a closed function. Let $\tilde{\mathscr{U}}$ be a closed family in X such that $C\in\tilde{\mathscr{U}}$, so C is a closed proper subset of X. Since X is Lindelöf, then C is a Lindelöf subset of X, hence f(C) is Lindelöf because f is continuous [10]. But (Y,σ) is an L-closed space, so f(C) is a closed subset of Y.

Proposition 3. *If a continuous function from a Lindelöf Hausdorff space to an L-closed space is closed, then every continuous onto function is homeomorphism.*

Proposition 4. *Being L-closed space is a topological property.*

Proof. Let (X,τ) be an L-closed space and (Y,σ) be any space, let h:$(X,\tau)\rightarrow(Y,\sigma)$ be homeomorphism. If A is a Lindelöf subset of X, then h(A) is Lindelöf because h is continuous. If X is an L-closed space and A is closed, then h(A) is closed because h is a closed function. Thus Y is an L-closed space.

Proposition 5. *Let (X,τ) and (Y,σ) be two spaces such that X is L-closed. If f:$(X,\tau)\rightarrow(Y,\sigma)$ is any function and $\{$(x,f(x)): x$\in$X$\}$ is a Lindelöf subset of $X\times Y$, then f is continuous.*

Proof. Let π_x and π_y be two projection functions. If $\pi_x'=\pi_{x_{|f}}$, π_x' and π_y are continuous surjective functions. If $\tilde{\mathscr{A}}=\{(x,\mathrm{f}(x)): x\in \mathrm{X}\}$. Since $\tilde{\mathscr{A}}$ is Lindelöf, any closed subset of f is Lindelöf , so π_x' is a closed projection function, i.e if $A\subseteq$f is a closed subset, so A is a Lindelöf subset of X because X is L-closed space. Now, f is defined on X, so π_x' is a onto function. Using this fact and the fact that π_x' is a closed projection function, we get that open set $v\subseteq$f we have $\pi_x'(v)$ is open in X.

Hence f $=\pi_y\circ(\pi_x')^{-1}$ is p-continuous.

Proposition 6. *If the topological spaces (X,τ) and (Y,σ) are L-closed such that either X or Y is regular, then $X\times Y$ is an L-closed space.*

Proof. If (X,τ) and (Y,σ) are L-closed spaces and let Y be regular. If F is a Lindelöf subset of X $\times$ Y. Let $(x_0,y_0)\in cl(F)$-F, so $(x_0,y_0)\notin[(\{x_0\}\times$ Y $]\cap F$ and $(\{x_0\}\times Y)\cap F$ is a closed subset of $X\times Y$ because Y is an L-closed space. Y is regular, so $\exists H$ an open subset containing y_0 such that (X $\times clH)\cap[(\{x_0\}\times$ Y$)\cap F$ $]=\phi$, so the projection function π_x((X $\times$ $cl_2H)\cap((\{x_0\}\times$ Y$)\cap F)$ is a closed subset of X because π_x is continuous [2]. X-[π_x(X $\times\, cl_2H)\cap F)\times$ (Y$\cap$(X $\times$ H))] is an open neighborhood of (x_0,y_0) disjoint from F, hence a subset F is closed in X $\times$ Y.

Proposition 7. *The product of finite number of p-regular pairwise L-closed spaces is pairwise L-closed.*

Proof. Let $\{$ (X_k,τ^k): k=1,2,...,n$\}$ be a family of finitely many regular L-closed spaces. Let X$=\prod_{k\in\mathbb{N}}$X$_k$. By the induction on k, for k=2 the result is given by 3.2.1. Suppose that this is true for k=n, we claim that it is true for k=n+1. $(\mathrm{X}_1\times \mathrm{X}_2\times\ldots\times\mathrm{X}_n)\times\mathrm{X}_{n+1}$ is homeomorphic to $\mathrm{X}_1\times\mathrm{X}_2\times\ldots\times\mathrm{X}_n\times$ X_{n+1} and $(\mathrm{X}_1\times\mathrm{X}_2\times\ldots\times\mathrm{X}_n)\times\mathrm{X}_{n+1}$ is L-closed space by the assumption and using the fact that the product of two regular L-closed spaces. Hence X is L-closed [9].

From the previous prepositions we conclude that the finite product of L-closed regular spaces is L-closed, however infinite product of L-closed regular spaces needs not to be L-closed [12] as the following example shows, for example, if we consider that X= $\{0,1\}$ is endowed with two topologies that are discrete, then X is regular L-closed [5]. Nevertheless, the countable product Y$=\prod_{\alpha\in\Lambda}X_\alpha$ is compact and infinite. Thus, a space Y cannot be L-closed.

Proposition 8. *Suppose that for the two spaces (X,τ) which is an L-closed space and (Y,σ) is a Lindelöf space, so the projection function π_x:$X\times Y\rightarrow X$ is Lindelöf.*

Proof. Consider a Lindelöf subset F of X, then F is closed since X is an L-closed space. In addtion, a subset F is L-closed, hence it is P-subset and $\pi_{x|F\times Y}$ is closed where $(\pi_{x|F\times Y})^{-1}$(X) is Lindelöf [13].

Proposition 9. *A space* (X,τ) *is a P-space if and only if for each Lindelöf space (Y,σ), the projection function π_x:* X $\times$ Y$\to$ X *is closed.*

Proof. $\Rightarrow$) By Kuratowski's Theorem concerning compact topological spaces [16]. If a subset F of X is a closed and X is a P-space, if a point $y \notin \pi_x(F \times \mathrm{Y})$, $y \in V$ where V is an open subset of Y, then (X$\times V$)$\cap F$=ϕ because X$\times\{y\}\subseteq$(X$\times$ Y)$\setminus F$ [1]. Hence, $\pi_x(F \times \mathrm{Y})$)$\cap$V=ϕ, that is, a subset $\pi_x(F)$ of X is closed [15].

Contrarily, each non-P-space is as follows, whenever π_x is closed, a space Y has to be countably compact, thus X is non-P-space and this fact isobtained by Hanai, 1962.

Example 1. Consider the topological spaces (X,τ) and ($\mathbb{R}$,σ) where X is a P-space and $\mathbb{R}$ is Lindelöf . The projection function π_x:X$\times\mathbb{R}$ $\to$X is not closed eventhough $\mathbb{R}$ is Lindelöf [10].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Al-labadi. M., Almuhur. E.,(2020), Planar of special idealization rings, WSEAS Transactions on Mathematics, 19(66), pp.606-609.
[2] Almuhur. E., Al-labadi. M., (2021), Pairwise Strongly Lindelöf, Pairwise Nearly, Almost and Weakly Lindelöf Bitopological Spaces, WSEAS Transactions on Mathematics, 20(16), pp.152-158.
[3] Engelking,(1977), General Topology, PWN-Polish Sci. Publ. Warszawa, Berlin.
[4] Ewert. J., (1986), On Pairwise Hausdorff Bitopological Spaces, Pedagogical University, Arciszewskiego. Poland, 22, pp.76-200.
[5] Fletcher. P., Hoyle. H.B., and Patty. C.W., (1969), The comparison of Topologies, Duke Mathematical Journal, 36, pp.325-331.
[6] Hdeib. H., and Pareek. C. M., (1986), On Spaces In Which Lindlof Sets are Closed, Q&A in General Topology,4, pp.67-72.
[7] Hdeib. H., (1979), Contribution to the theory of [n,m]-compact, Paracompact and normal spaces, Ph.D. Thesis SUNY at Buffalo.
[8] Jankovic. D., Ellen. M., (1999), On Some Generalizations of L-closed spaces, Acta Mathematica Universitatis Comenianae, 2, pp.345-353.
[9] Kim. W., (1968), Pairwise Compactness, Publicationes Mathematicae Debrecen, 15, pp.87-90.
[10] Misra. A., (1972), Topological View of P-Spaces, General Topology and Its Applications, 2, pp.349-362.
[11] Milovančević. D., (1985), Property Between Compact and Strongly Countably Compact, Publications De L'Institut Mathema tique, 38, pp.193-201.
[12] Mukharjee. A., (2013), Some New Bitopological Notions, Publications De L'Institut Mathématique, 93,pp.165–172.
[13] Omer. N., Hdeib. H., Almuhur. E., Al-labad. M., (2021), ON P_2-C-CLOSED SPACE IN BITOPOLOGICAL SPACE, Advances in Mathematics: Scientific Journal, 10(3), pp.1839-1843.
[14] Pahk. H., and Choi. D., (1971), Notes on pairwise compactness, Kyungpook Mathimatical Journal, 11, pp.45-52.
[15] Patty. W., (1967), Bitopological spaces, Duke Mathimaticla Journal, 34, pp.387–391.
[16] Weston. D., (1957), On the comparison of Topologies, Journal of London Mathimatical Society, 32, pp.342-354.
[17] Henriksen. M., Woods, (1988), Weak P-space and L-closed spaces, Q&A in General Topology, 6, pp.201-206.