NTMSCI **9 Special Issue**, No. 1, 118-123 (2021)

New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2021.438

# On degenerate truncated Frobenius-Euler polynomials

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Received: 13 June 2021, Accepted: 16 June 2021
Published online: 12 July2021.

**Abstract:** In this study, we consider the truncated degenerate Frobenius-Euler polynomials. Then we examine diverse properties and formulas covering addition formulas, correlations and derivation property. Then, we derive some interesting implicit summation formulas.

**Keywords:** Degenerate exponential function, truncated exponential function, Frobenius-Euler polynomials, exponential generating function.

## 1 Introduction

Along this paper, the usual notations $\mathscr{N}$, $\mathscr{N}_0$, $\mathscr{R}$ and $\mathscr{C}$, are referred to the set of all natural numbers, the set of all non-negative integers, the set of all real numbers and the set of all complex numbers, respectively.

The truncated form of the exponential polynomials $e_n(t)$ are the first $(n+1)$ terms of the Taylor series for $e^t$ (*cf.* [2]) at $t=0$, that is,

$$e_n(t)=\sum_{k=0}^{n}\frac{t^k}{k!}. \tag{1}$$

One can see [2] to get the detailed information about $e_n(t)$.

For $\lambda\in\mathscr{C}$, the $\lambda$-falling factorial $(t)_{n,\lambda}$ is defined by $(t)_{n,\lambda}=t(t-\lambda)(t-2\lambda)\cdots(t-(n-1)\lambda)$ for $n\in\mathscr{N}$ with $(t)_{0,\lambda}=1$, *cf.* [1,3,5-8]. In the case $\lambda=1$, the $\lambda$-falling factorial becomes to the usual falling factorial given by $(t)_{n,1}:=(t)_n=t(t-1)\cdots(t-n+1)$ with $(t)_{0,1}=1$.

Let $\lambda\in\mathscr{R}/\{0\}$. The degenerate form of the exponential function $e^t_\lambda(t)$ is defined by (*cf.* [1,3,5-8])

$$e^{\omega}_{\lambda}(t)=(1+\lambda t)^{\frac{\omega}{\lambda}} \text{ and } e^{1}_{\lambda}(t):=e_{\lambda}(t). \tag{2}$$

We note that $\lim_{\lambda\to0}e^{\omega}_{\lambda}(t)=e^{\omega t}$. From (2), we attain

$$e^{\omega}_{\lambda}(t)=\sum_{n=0}^{\infty}(\omega)_{n,\lambda}\frac{t^n}{n!}. \tag{3}$$


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*Proof.* The $\Delta_\lambda$ difference operator of a function is defined by (see [3])

$$\Delta_\lambda f(x) = \frac{1}{\lambda}(f(x+\lambda) - f(x)), \quad \alpha \neq 0. \tag{4}$$

The degenerate truncated form of the exponential polynomials (also called the $Detr$-exponential polynomials) are considereed as the first $(n+1)$ terms of the Mac Laurin series expansion of $e_\lambda(t)$ in (3) (*cf.* [3]):

$$e_{n,\lambda}(t) = \sum_{k=0}^{n} (1)_{k,\lambda} \frac{t^k}{k!}. \tag{5}$$

Also, when $\lambda \to 0$, the polynomials $e_{n,\lambda}(t)$ (5) become the polynomials $e_n(t)$ in (1). To get more detailed information about the $Detr$-exponential polynomials and their properties, see [3].

The Stirling numbers $S_2(n,k)$ and polynomials $S_2(n,k:\omega)$ of the second kind are given as follows (*cf.* [1,3,6-9]):

$$\sum_{n=0}^{\infty} S_2(n,k) \frac{t^n}{n!} = \frac{(e^t-1)^k}{k!} \text{ and } \sum_{n=0}^{\infty} S_2(n,k:\omega) \frac{t^n}{n!} = \frac{(e^t-1)^k}{k!} e^{t\omega}. \tag{6}$$

The degenerate form of the Stirling polynomials of the second kind are given below (*cf.* [1,3,6,7]):

$$\sum_{n=0}^{\infty} S_{2,\lambda}(n,k:\omega) \frac{t^n}{n!} = \frac{(e_\lambda(t)-1)^k}{k!} e_\lambda^{\omega}(t). \tag{7}$$

The degenerate truncated form of the Stirling polynomials of the second kind are given as follows (*cf.* [3]):

$$\sum_{n=0}^{\infty} S_{2,m;\lambda}(n,k:\omega) \frac{t^n}{n!} = \frac{\left(e_\lambda(t) - 1 - e_{m-1,\lambda}(t)\right)^k}{k!} e_\lambda^{\omega}(t). \tag{8}$$

## 2 On degenerate truncated Frobenius-Euler polynomials

In this section, we introduce the truncated degenerate Frobenius-Euler polynomials and investigated multifarious correlations and formulas including summation formulas, derivation rules and correlation with the degenerate Stirling numbers of the second kind.

Let $u(\neq 1) \in \mathscr{C}$ is an algebraic number. The classical Frobenius-Euler $H_n(u,x)$ polynomials (*cf.* [4,8,9]) are given as follows:

$$\sum_{n=0}^{\infty} H_n(u,x) \frac{t^n}{n!} = \frac{1-u}{e^t-u} e^{xt},$$

The usual degenerate Frobenius-Euler $H_{n,\lambda}(u,x)$ polynomials are defined as follows (*cf.* [8]):

$$\sum_{n=0}^{\infty} H_{n,\lambda}(u,x) \frac{t^n}{n!} = \frac{1-u}{e_\lambda(t)-u} e_\lambda^{x}(t).$$

We now introduce the degenerate truncated forms of the Frobenius-Euler polynomials as follows.

**Definition 1.** *Let $x$ be an independent variable. The degenerate truncated Frobenius-Euler polynomials are defined by* *the following exponential generating function:*

$$\sum_{n=0}^{\infty} H_{m,n,\lambda}(u,x)\frac{t^n}{n!} = \frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t) - u - e_{m-1,\lambda}(t)} e_\lambda^x(t). \tag{9}$$

We choose to call the $Detr$-Frobenius-Euler polynomials besides the degenerate truncated Frobenius-Euler polynomials.

When $x = 0$ in Definition 1, the $Detr$-Frobenius-Euler polynomials $H_{m,n,\lambda}(u,x)$ reduce to the corresponding numbers called the $Detr$-Frobenius-Euler numbers denoted by $H_{m,n,\lambda}(u)$:

$$\sum_{n=0}^{\infty} H_{m,n,\lambda}(u)\frac{t^n}{n!} = \frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t) - u - e_{m-1,\lambda}(t)}. \tag{10}$$

We now perform to derive some properties of the aforementioned polynomials and we first give the following correlation.

**Theorem 1.** *The following summation formula holds:*

$$H_{m,n,\lambda}(u,x) = \sum_{k=0}^{n} \binom{n}{k} (x)_{k,\lambda} H_{m,n-k,\lambda}(u). \tag{11}$$

*Proof.* In view of the Definition 1 and using (10), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} H_{m,n,\lambda}(u,x)\frac{t^n}{n!} &= \frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t) - u - e_{m-1,\lambda}(t)} e_\lambda^x(t) \\
&= \sum_{n=0}^{\infty} H_{m,n,\lambda}(u)\frac{t^n}{n!} \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} (x)_{k,\lambda} H_{m,n-k,\lambda}(u)\frac{t^n}{n!},
\end{aligned}$$

which completes the proof of the Theorem 1. An addition formula is presented by the following theorem.

**Theorem 2.** *The following relationship is valid.*

$$H_{m,n,\lambda}(u,x_1+x_2) = \sum_{l=0}^{n} \binom{n}{l} (x_2)_{n-l,\lambda} H_{m,l,\lambda}(u,x_1). \tag{12}$$

*Proof.* They are similar to Theorem 1. So, we omit them.

**Corollary 1.** *The following explicit relation holds:*

$$H_{m,n,\lambda}(u,x+1) = \sum_{l=0}^{n} \binom{n}{l} (1)_{n-l,\lambda} H_{m,l,\lambda}(u,x). \tag{13}$$

**Theorem 3.** *The difference operator formula for the $Detr$-Frobenius-Euler polynomials*

$$\Delta_\lambda E_{m,n,\lambda}(x) = nE_{m,n-1,\lambda}(x) \tag{14}$$

holds for $m,n \in \mathcal{N}_0$.

*Proof.* By applying difference operator $\Delta_\lambda$ (4) to both sides of the formula (9), we attain

$$\sum_{n=0}^{\infty} \Delta_\lambda H_{m,n,\lambda}(u,x)\frac{t^n}{n!} = \frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t)-u-e_{m-1,\lambda}(t)}\Delta_\lambda e_\lambda^x(t)$$

$$= \frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t)-u-e_{m-1,\lambda}(t)}e_\lambda^x(t)t = \sum_{n=0}^{\infty} H_{m,n,\lambda}(u,x)\frac{t^{n+1}}{n!},$$

which gives the claimed difference property in (14).

The $Detr$-Frobenius-Euler polynomials satisfy the following derivative property.

**Theorem 4.** *We have*

$$\frac{d}{dx}H_{m,n;\lambda}(u,x) = n!\sum_{p=1}^{\infty} H_{m,n-p;\lambda}(u,x)\frac{(-1)^{p+1}}{(n-p)!p}\lambda^{p-1}. \tag{15}$$

*Proof.* By applying the derivative operator $d/dx$ with respect to $x$ to both sides of the formula (9), we then derive

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}H_{m,n;\lambda}(u,x)\frac{t^n}{n!} &= \frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t)-u-e_{m-1,\lambda}(t)}\frac{d}{dx}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t)-u-e_{m-1,\lambda}(t)}(1+\lambda t)^{\frac{x}{\lambda}}\ln(1+\lambda t)^{\frac{1}{\lambda}} \\ &= \sum_{n=0}^{\infty} H_{m,n;\lambda}(u,x)\frac{t^n}{n!}\sum_{u=1}^{\infty}\frac{(-1)^{u+1}}{u}\lambda^{u-1}t^u\end{aligned}$$

which gives the assertion in (15).

A summation identity for $H_{m,n;\lambda}(u,x)$ is presented in the following theorem.

**Theorem 5.** *The following recurrence formula*

$$H_{m+1,n,\lambda}(u,x) = n\frac{1-m\lambda}{m+1}H_{m,n-1;\lambda}(u,x) + \frac{1}{1-u}\sum_{k=0}^{n}\binom{n}{k}H_{m,n-k;\lambda}(u)H_{m+1,k;\lambda}(u,x) \tag{16}$$

is valid for $n,m \in \mathcal{N}_0$.

*Proof.* From Definition 1, we can write

$$(1-u)\frac{t^{m+1}}{(m+1)!}(1)_{m+1,\lambda}e_\lambda^x(t) = \left(e_\lambda(t)-u-e_{m,\lambda}(t)\right)\sum_{n=0}^{\infty}H_{m+1,n,\lambda}(u,x)\frac{t^n}{n!}$$

$$= \left(e_\lambda(t)-u-e_{m-1,\lambda}(t)\right)\sum_{n=0}^{\infty}H_{m+1,n,\lambda}(u,x)\frac{t^n}{n!}(1)_{m,\lambda}\frac{t^m}{m!} - \sum_{n=0}^{\infty}H_{m+1,n,\lambda}(u,x)\frac{t^n}{n!}$$

Hence, we observe that

$$\frac{(1-u)\frac{t^{m+1}}{(m+1)!}(1)_{m+1,\lambda}}{e_\lambda(t)-u-e_{m-1,\lambda}(t)}e_\lambda^x(t) = \sum_{n=0}^{\infty}H_{m+1,n,\lambda}(u,x)\frac{t^n}{n!}$$

$$-\frac{1}{1-u}\frac{(1-u)(1)_{m,\lambda}\frac{t^m}{m!}}{e_\lambda(t)-1-e_{m-1,\lambda}(t)}\sum_{n=0}^{\infty}H_{m+1,n,\lambda}(u,x)\frac{t^n}{n!}$$

which means the asserted result (16). We also provide a relationship as follows.

**Theorem 6.** *The following identity*

$$H_{m,n,\lambda}(u,x)=\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{l}H_{m,n-l,\lambda}(u)S_{2;\lambda}(l,k:-k)(x)^{(k)} \tag{17}$$

is valid for $n,m\in\mathcal{N}_0$.

*Proof.* By the Definition 1 and utilizing the formulae and (10), we attain

$$\begin{aligned}\sum_{n=0}^{\infty}H_{m,n,\lambda}(u,x)\frac{t^n}{n!}&=\frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t)-u-e_{m-1,\lambda}(t)}\left(e_\lambda^{-1}(t)-1+1\right)^x\\&=\frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t)-u-e_{m-1,\lambda}(t)}\sum_{k=0}^{\infty}\binom{x+k-1}{k}\left(1-e_\lambda^{-1}(t)\right)^k\\&=\frac{(1-u)\frac{t^m}{m!}(1)_{m,\lambda}}{e_\lambda(t)-u-e_{m-1,\lambda}(t)}\sum_{k=0}^{\infty}\binom{x+k-1}{k}\frac{(e_\lambda(t)-1)^k}{k!}e_\lambda^{-k}(t)k!\\&=\sum_{k=0}^{\infty}(x)^{(k)}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}H_{m,n-l,\lambda}(u)S_{2;\lambda}(l,k:-k)\right)\frac{t^n}{n!},\end{aligned}$$

which gives the assertion (17).

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

## References

[1] L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers. Utilitas Math. 15 (1975) 51-88.
[2] G. Dattoli, C. Ceserano, D. Sacchetti, A note on truncated polynomials. Appl. Math. Comput. 134 (2003) 595-605.
[3] U. Duran, M. Acikgoz, On degenerate truncated special polynomials, Mathematics, 2020, 8(1), 144.
[4] T. Kim, B. Lee, Some identities of the Frobenius-Euler polynomials, Volume 2009 Article ID 639439, doi:10.1155/2009/639439.
[5] Kim, D. S.; Kim, T.; Lee, H. A note on degenerate Euler and Bernoulli polynomials of complex variable. Symmetry 2019, 11, 1168.
[6] T. Kim, Y. Yao, D.S. Kim, G.-W. Jang, Degenerate -Stirling numbers and -Bell polynomials. Russ. J. Math. Phys. 25 (2018) 44-58.
[7] T. Kim, A note on degenerate Stirling polynomials of the second kind. Proc. Jangjeon Math. Soc. 20 (2017) 319-331.
[8] T. Kim, H. I. Kwon, J.-J. Seo, On the degenerate Forbenius-Euler polynomials, arXiv:1507.04846v1 [math.NT].
[9] H. . Srivastava, S. Araci, W.A. Khan, M. Acikgoz, A note on the truncated-exponential based Apostol-type polynomials, Symmetry. 11 (2019) 538.

[10] B. Y. Yasar, M. A. Ozarslan, Frobenius-Euler and Frobenius-Genocchi polynomials and their differential equations, New Trends in Mathematical Sciences, 3(2) (2015) 172-180.