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Coefficients of singularities for boundary value problems governed by the Lamé system (elasticity) in a plane sector

Razika Boufenouche

LMPA, University of Mohammed Seddik Ben Yahia, Jijel, Algeria

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Abstract: This paper represents the solution to boundary value problems governed by the Lamé system (elasticity) in a plane sector. By using appropriate Betti function, we establish a biorthogonality relation between the terms of the series, which allow us to calculate the coefficients of singularities in the important case crack.

Keywords: Sector, Crack, Singularity, Lamé system(elasticity), Series.

1 Introduction

Let S be the truncated plane sector of angle $\omega \leq 2\pi$, and radius ρ (ρ positive fixed) defined by:

$$S = \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2, 0 < r < \rho, 0 < \theta < \omega \right\} \tag{1.1}$$

and γ the circular boundary part defined by:

$$\gamma = \left\{ (\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2, 0 < \theta < \omega \right\} \tag{1.2}$$

We are interested in the study of a function u, belonging to the Sobolev space $\left[H^1(S)\right]^2$, and being the solution of

$$\begin{cases} Lu = \Delta u + v_0 \nabla (divu) = 0, \text{ in } S \\ \Sigma(u).\eta = 0, \qquad \text{for } \theta = 0 \\ \begin{cases} u.\tau = 0 \\ (\Sigma(u).\eta).\eta = 0 \end{cases}, \text{ for } \theta = \omega \end{cases} \tag{1.3}$$

where $v_0 = (1-2\nu)^{-1}$, here ν is a real number $\left(0 < \nu < \frac{1}{2}\right)$ called Poisson coefficient, η is the outward unit vector normal to γ and τ is the tangential vector. We show that the solutions u of this problem can be written in a series of the form:

$$u(r,\theta) = \sum_{\alpha \in E} c_\alpha r^\alpha v_\alpha(\theta). \tag{1.4}$$

Here E stands for the set of solutions of the equation in a complex variable $\alpha(v_0)$:

$$\sin 2\alpha\omega = \alpha \sin 2\omega, \text{Re } \alpha > 0. \tag{1.5}$$

* Corresponding author e-mail: r_boufenouche@yahoo.fr

We will adapt the technique used in [6, 7], for the Dirichlet condition by the the Betti formula instead of a Green formula for the domain S. To compute the coefficients c_α of the singularities which can occur in the solutions, we will focus on the important case of the crack $(\omega = 2\pi)$. The calculation in that case are more explicit and give the known results for the Laplacien as just a particular case.

2 Transformation of the problem

Replacing u by $r^\alpha v_\alpha(\theta)$ in problem (1.3) and making the change of funtions:

$$\begin{cases} w_1(\theta) = \cos\theta v_1(\theta) + \sin\theta v_2(\theta); \\ w_2(\theta) = -\sin\theta v_1(\theta) + \cos\theta v_2(\theta) \end{cases} \tag{2.1}$$

leads us to the system:

$$L_\alpha(w) = \begin{cases} w_1''(\theta) + (\nu_0+1)\left(\alpha^2-1\right) w_1(\theta) + \rho_0 w_2'(\theta) = 0; \\ (\nu_0+1) w_2''(\theta) + \left(\alpha^2-1\right) w_2(\theta) + \rho_1 w_1'(\theta) = 0. \\ w_1'(0) + (\alpha-1) w_2(0) = (\nu_0(\alpha+1) - (\alpha-1)) w_1(0) + (\nu_0+1) w_2'(0) = 0; \\ w_1(0) = w_2'(0) = 0. \end{cases} \tag{2.2}$$

where:

$$\rho_0 = \nu_0(\alpha-1) - 2 \text{ and } \rho_1 = \nu_0(\alpha+1) + 2.$$

A relation similar to orthogonality obtained for this operator is given in the following theorem.

Theorem 1. *Let w_α and w_β be solutions of $(2,2)$ with α and β solutions of $(1,5)$.Then, for $\alpha \neq \overline{\beta}$, we have*

$$[w_\alpha, w_\beta] = \int_0^\omega \left[\left[\frac{1}{\left(\overline{\beta}-\alpha\right)} \nu_0\left(w_{2,\beta}', w_{1,\beta}'\right) + (\nu_0+1) w_{1,\beta}, w_{2,\beta}\right)\right] \begin{pmatrix} \overline{w}_{1,\alpha} \\ \overline{w}_{2,\alpha} \end{pmatrix} \Bigg] d\theta = 0. \tag{2.3}$$

Corollary 1. *Let w_α and w_β be solutions of (2.2) with α and β solutions of (1.5). Suppose in addition that:*

$$\int_0^\omega \left(w_{2,\alpha}', w_{2,\alpha}'\right) \begin{pmatrix} \overline{w}_{1,\alpha} \\ \overline{w}_{2,\alpha} \end{pmatrix} d\theta = 0, \tag{2.4}$$

and $\alpha \neq \overline{\beta}$, then:

$$[w_\alpha, w_\beta] = \int_0^\omega [(1+\nu_0) w_{1,\alpha}, w_{2,\alpha})] \begin{pmatrix} \overline{w}_{1,\beta} \\ \overline{w}_{2,\beta} \end{pmatrix} d\theta = 0. \tag{2.5}$$

Remark. For $u_\alpha = r^\alpha \varphi_\alpha$. Suppose that

$$T u_\alpha = r^{\alpha-1} \begin{pmatrix} (1+\nu_0)\varphi_{1,\alpha} \\ \varphi_{2,\alpha} \end{pmatrix}$$

where T be is a linear operator. From the Corollary 2.1 and Remark 2.1, we deduce the following result.

Corollary 2. *By the Corollary* 2.1 *If $\alpha \neq \overline{\beta}$, we have:*

$$\int_\gamma \left(T u_\alpha . \overline{u}_\beta + u_\alpha . T\overline{u}_\beta\right) d\sigma = 0. \tag{2.6}$$

Corollary 3. *Suppose that* $u = \sum\limits_{\alpha \in E} c_\alpha r^\alpha \varphi_\alpha$ *is uniformly convergent dans* $\overline{S}$*. Si* $\left[\varphi_{\overline{\beta}}, \varphi_\beta\right] \neq 0$ *then:*

$$c_{\overline{\beta}} = \frac{1}{2}\rho^{-2\overline{\beta}} \frac{\int\limits_\gamma \left(Tu_\alpha.\overline{u}_\beta + u_\alpha.T\overline{u}_\beta\right) d\sigma}{\left[\varphi_{\overline{\beta}}, \varphi_\beta\right]}.$$

Remark. The technique we develop for the study of the trigonometric series is based on Theorem 1 and Corollarye 1. We study the following trigonometric series in the particular case of singular domains, in the applications. Moreover in this case the solutions of (1.5) are explicitly known and we have

$$E = \left\{\frac{k}{2},\ k \in \mathbb{N}^*, k > 2\right\}.$$

3 Formula for the coefficients in the crack case.

In this framework we assume that the solution u of (1.3) decompose in two parts with respect to θ

$$u = \sum_{\alpha \in E} \left(c_\alpha u_\alpha + d_\alpha v_\alpha\right),\ u_\alpha\left(r, \theta\right) = r^\alpha \varphi_\alpha\left(\theta\right),\ v_\alpha\left(r, \theta\right) = r^\alpha \psi_\alpha\left(\theta\right), \tag{3.1}$$

the solutions φ_α and ψ_α in terms of θ are the functions:

$$\varphi_\alpha\left(\theta\right) = \begin{pmatrix} \alpha\nu_0 \cos\left(\alpha - 2\right)\theta - \left(\nu_0\left(\alpha + 2\right) + 2\right)\cos\alpha\theta \\ -\alpha\nu_0 \sin\left(\alpha - 2\right)\theta + \left(\nu_0\alpha - 2\right)\sin\alpha\theta \end{pmatrix}. \tag{3.2}$$

$$\psi_\alpha\left(\theta\right) = \begin{pmatrix} \alpha\nu_0 \sin\left(\alpha - 2\right)\theta - \left(\nu_0\alpha + 2\right)\sin\alpha\theta \\ \alpha\nu_0 \cos\left(\alpha - 2\right)\theta - \left(\nu_0\left(\alpha - 2\right) - 2\right)\cos\alpha\theta \end{pmatrix}, 0 < \theta < \omega. \tag{3.3}$$

3.1 Study of the first part

In this part the expression φ_α is given by (3.2). After calculation, we obtain that

$$\left[\varphi_\beta, \varphi_\beta\right] = \pi\left[\alpha^2\nu_0^2\left(\nu_0 + 2\right) + \left(\nu_0 + 1\right)\left(\nu_0\left(\alpha + 2\right) + 2\right)^2 + \left(\nu_0\alpha - 2\right)^2\right].$$

3.2 Study of the second part

The expression ψ_α is given by (3.3) and

$$\left[\psi_\beta, \psi_\beta\right] = \pi\left[\alpha^2\nu_0^2\left(\nu_0 + 2\right) + \left(\nu_0\alpha + 2\right)^2 + \left(\nu_0\left(\alpha - 2\right) - 2\right)^2\right].$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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