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A 3-step block hybrid backward differentiation formulae (bhbdf) for the solution of general second order ordinary differential equation

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Abstract: In this paper, the block hybrid Backward Differentiation formulae (BHBDF) for the step number k=3 is developed using power series as basis function for the solution of general second order ordinary differential equation. The idea of interpolation and collocation of the power series at some selected grid and off- grid points gave rise to continuous schemes which were further evaluated at those points to produce discrete schemes combined together to form block methods. Numerical problems were solved with the proposed methods and were found to perform effectively.

Keywords: Block, Hybrid, Backward differentiation formulae, Ordinary differential equation.

1 Introduction

Numerical analyst are usually faced with the challenge of obtaining starting or initial values for Linear Multistep Methods (LMM) when step number $k \geq 2$in solving differential equations numerically. Prior before now, one step methods like Runge-Kutta or trapezoidal methods were used to obtain the starting values for such methods. The hybrid method is not exempted from this problem as it shares the same standard methods. In addition to this, the need for special predictors to predict the off-grid values in hybrid forms of the LMM. The discrete schemes obtained from the continuous formulation of k-step block hybrid backward differentiation formulae can be used in block form to obtain the block solution. This approach circumvents both the non self starting property and calls for special predictors to predict the off-grid values in the hybrids forms of the block hybrid backward differentiation formulae.

Methodology

We present the derivation of Block Hybrid Backward Differentiation Formula (HBDF) for solving some classes of second-order ordinary differential equations of the form

$$\frac{d^2y(x)}{dx^2} = f\left(x, y, \frac{dy(x)}{dx}\right) \tag{1}$$

coupled with appropriate initial conditions

$$y(x_0) = \varphi_1, \frac{dy(x_0)}{dx} = \varphi_2 \tag{2}$$

where f is a continuous function such that $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$, x_0 is the initial point, $y \in \mathbb{R}$ is an $n-$dimensional vector, x is a scalar variable, φ_1 and φ_2 are the initial values.

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In this research, we seek to develop numerical schemes in the form of HBDF as:

$$Y(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+\alpha_v(x)y_{n+v}+h^2\beta_k(x)f_{n+k} \tag{3}$$

where h is the chosen step size and $\alpha_j(x):\ j=0,1,2,\ldots,\ k,\ \alpha_v(x),\ \beta_k(x)$ are unknown continuous coefficients to be determined.For Backward Differentiation Formula, we note that $\alpha_k=1$ and $\beta_k\neq 0$. In this study, we will derive Block Hybrid Backward Differentiation (BHBDF) for the step numbers $k=3$ of the proposed method using power series function as the basis function.

We seek an approximation of the form;

$$Y(x)=\sum_{j=0}^{t+c-1}p_jx^j \tag{4}$$

where t is the interpolation points, c is the collocation points and p_j are unknown coefficients to be determined. Then, we take

$$(x)=y_{n+j},\ j=0,1,2,\ldots,\ k-1 \tag{5}$$

$$Y''(x_{n+k})=f_{n+k} \tag{6}$$

To derive a 3SBHBDF, we take $t=5,\ c=1$ and $x\in[x_n,\ x_{n+3}]$. Therefore, (4) becomes;

$$Y(x)=\ p_0+p_1x+p_2x^2+p_3x^3+p_4x^4+p_5x^5 \tag{7}$$

Interpolating (5) at $x_{n+i};i=0,\ 1,\ 2,\ \frac{5}{2},\frac{11}{4}$ and collocate (6) at $x_{n+i};i=3$. This results in a system of equations;

$$D\psi=Y \tag{8}$$

where $\psi=\left(\alpha_0,\alpha_1,\alpha_2,\alpha_{\frac{5}{2}},\alpha_{\frac{11}{4}},\beta_3\right)^T,\ Y=\left(y_n,\ y_{n+1},y_{n+2},\ y_{n+\frac{5}{2}},y_{n+\frac{11}{4}},\ f_{n+3}\right)^T$ and the matrix D of the proposed method is expressed as

$$D=\begin{bmatrix} & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5\\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5\\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5\\ 1 & x_{n+\frac{5}{2}} & x_{n+\frac{5}{2}}^2 & x_{n+\frac{5}{2}}^3 & x_{n+\frac{5}{2}}^4 & x_{n+\frac{5}{2}}^5\\ 1 & x_{n+\frac{11}{4}} & x_{n+\frac{11}{4}}^2 & x_{n+\frac{11}{4}}^3 & x_{n+\frac{11}{4}}^4 & x_{n+\frac{11}{4}}^5\\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2} & 20x_{n+2}\end{bmatrix}$$

We proceed by solving (8) using matrix inversion method with the aid of Maple 2017 software to obtain the continuous coefficients;

$$\left.\begin{aligned}
\alpha_0&=-\frac{7}{330}\frac{x^5}{h^5}+\frac{109}{440}\frac{x^4}{h^4}-\frac{2963}{2640}\frac{x^3}{h^3}+\frac{2157}{880}\frac{x^2}{h^2}-\frac{3373}{1320}\frac{x}{h}+1\\
\alpha_1&=\frac{7}{45}\frac{x^5}{h^5}-\frac{233}{140}\frac{x^4}{h^4}+\frac{16613}{2520}\frac{x^3}{h^3}-\frac{3209}{280}\frac{x^2}{h^2}+\frac{1859}{252}\frac{x}{h}\\
\alpha_2&=-\frac{79}{90}\frac{x^5}{h^5}+\frac{343}{40}\frac{x^4}{h^4}-\frac{21563}{720}\frac{x^3}{h^3}+\frac{3479}{80}\frac{x^2}{h^2}-\frac{1529}{72}\frac{x}{h}\\
\alpha_{\frac{5}{2}}&=\frac{16}{9}\frac{x^5}{h^5}-\frac{84}{5}\frac{x^4}{h^4}+\frac{2522}{45}\frac{x^3}{h^3}-\frac{386}{5}\frac{x^2}{h^2}+\frac{1628}{45}\frac{x}{h}\\
\alpha_{\frac{11}{4}}&=-\frac{512}{495}\frac{x^5}{h^5}+\frac{3712}{385}\frac{x^4}{h^4}-\frac{109376}{3465}\frac{x^3}{h^3}+\frac{16448}{385}\frac{x^2}{h^2}-\frac{13696}{693}\frac{x}{h}\\
\beta_3&=\frac{1}{30}\frac{x^5}{h^3}-\frac{11}{40}\frac{x^4}{h^2}+\frac{197}{240}\frac{x^3}{h}-\frac{83}{80}x^2+\frac{11}{24}xh
\end{aligned}\right\} \tag{9}$$

The values of the continuous coefficients (9) are then substituted in to the proposed method in (3) to obtain

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_{\frac{5}{2}}(x)y_{n+\frac{5}{2}} + \alpha_{\frac{11}{4}}(x)y_{n+\frac{11}{4}} + \beta_3(x)f_{n+3} \tag{10}$$

Expressing (10) further gives the continuous form of the 3SHBDF with 2-step off- step interpolation point as

$$\begin{aligned} y(x) = &\left(-\frac{7}{330}\frac{x^5}{h^5} + \frac{109}{440}\frac{x^4}{h^4} - \frac{2963}{2640}\frac{x^3}{h^3} + \frac{2157}{880}\frac{x^2}{h^2} - \frac{3373}{1320}\frac{x}{h} + 1\right)y_n \\ &+ \left(\frac{7}{45}\frac{x^5}{h^5} - \frac{233}{140}\frac{x^4}{h^4} + \frac{16613}{2520}\frac{x^3}{h^3} - \frac{3209}{280}\frac{x^2}{h^2} + \frac{1859}{252}\frac{x}{h}\right)y_{n+1} \\ &+ \left(-\frac{79}{90}\frac{x^5}{h^5} + \frac{343}{40}\frac{x^4}{h^4} - \frac{21563}{720}\frac{x^3}{h^3} + \frac{3479}{80}\frac{x^2}{h^2} - \frac{1529}{72}\frac{x}{h}\right)y_{n+2} \\ &+ \left(\frac{16}{9}\frac{x^5}{h^5} - \frac{84}{5}\frac{x^4}{h^4} + \frac{2522}{45}\frac{x^3}{h^3} - \frac{386}{5}\frac{x^2}{h^2} + \frac{1628}{45}\frac{x}{h}\right)y_{n+\frac{5}{2}} \\ &+ \left(-\frac{512}{495}\frac{x^5}{h^5} + \frac{3712}{385}\frac{x^4}{h^4} - \frac{109376}{3465}\frac{x^3}{h^3} + \frac{16448}{385}\frac{x^2}{h^2} - \frac{13696}{693}\frac{x}{h}\right)y_{n+\frac{11}{4}} \\ &+ \left(\frac{1}{30}\frac{x^5}{h^3} - \frac{11}{40}\frac{x^4}{h^2} + \frac{197}{240}\frac{x^3}{h} - \frac{83}{80}x^2 + \frac{11}{24}xh\right)f_{n+3} \end{aligned} \tag{11}$$

Evaluating (11) at $x = x_{n+3}$ gives the discrete scheme as

$$y_{n+3} = \frac{1}{440}y_n - \frac{11}{420}y_{n+1} + \frac{41}{120}y_{n+2} - \frac{28}{15}y_{n+\frac{5}{2}} + \frac{2944}{1155}y_{n+\frac{11}{4}} + \frac{1}{40}h^2 f_{n+3} \tag{12}$$

To obtain the sufficient schemes required, we obtain the first derivative of (11) and evaluate the continuous function at $x = x_n, x = x_{n+1}, x = x_{n+2}, x = x_{n+\frac{5}{2}}, x = x_{n+\frac{11}{4}} x = x_{n+3}$ to obtain;

$$\left.\begin{aligned} hz_n &= -\tfrac{3373}{1320}y_n + \tfrac{1859}{252}y_{n+1} - \tfrac{1529}{72}y_{n+2} + \tfrac{1628}{45}y_{n+\frac{5}{2}} - \tfrac{13696}{693}y_{n+\frac{11}{4}} + \tfrac{11}{24}h^2 f_{n+3} \\ hz_{n+1} &= -\tfrac{119}{880}y_n - \tfrac{461}{280}y_{n+1} + \tfrac{1393}{240}y_{n+2} - \tfrac{42}{5}y_{n+\frac{5}{2}} + \tfrac{5056}{1155}y_{n+\frac{11}{4}} - \tfrac{7}{80}h^2 f_{n+3} \\ hz_{n+2} &= \tfrac{1}{88}y_n - \tfrac{71}{420}y_{n+1} - \tfrac{299}{120}y_{n+2} + \tfrac{68}{15}y_{n+\frac{5}{2}} - \tfrac{2176}{1155}y_{n+\frac{11}{4}} + \tfrac{1}{40}h^2 f_{n+3} \\ hz_{n+\frac{5}{2}} &= -\tfrac{13}{3520}y_n + \tfrac{31}{672}y_{n+1} - \tfrac{161}{192}y_{n+2} - \tfrac{53}{30}y_{n+\frac{5}{2}} + \tfrac{2054}{385}y_{n+\frac{11}{4}} + \tfrac{77}{2560}h^2 f_{n+3} \\ hz_{n+\frac{11}{4}} &= \tfrac{133}{28160}y_n - \tfrac{1507}{26880}y_{n+1} + \tfrac{2079}{2560}y_{n+2} - \tfrac{1463}{240}y_{n+\frac{5}{2}} + \tfrac{2054}{385}y_{n+\frac{11}{4}} + \tfrac{77}{2560}h^2 f_{n+3} \\ hz_{n+3} &= \tfrac{31}{2640}y_n - \tfrac{337}{2520}y_{n+1} + \tfrac{1207}{720}y_{n+2} - \tfrac{74}{9}y_{n+\frac{5}{2}} + \tfrac{23104}{3465}y_{n+\frac{11}{4}} + \tfrac{47}{240}h^2 f_{n+3} \end{aligned}\right\} \tag{13}$$

where z is the first derivative of y. We further obtain the second derivatives of (11), thereafter, evaluating at $x = x_{n+\frac{5}{2}}, x = x_{n+\frac{11}{4}}, x = x_{n+2}$ to obtain;

$$\left.\begin{aligned} y_{n+\frac{5}{2}} &= \tfrac{141}{143968}y_n - \tfrac{617}{45808}y_{n+1} + \tfrac{4727}{13088}y_{n+2} + \tfrac{20512}{31493}y_{n+\frac{11}{4}} - \tfrac{45}{818}h^2 f_{n+\frac{5}{2}} + \tfrac{21}{13088}h^2 f_{n+3} \\ y_{n+\frac{11}{4}} &= -\tfrac{4641}{1037312}y_n + \tfrac{26521}{518656}y_{n+1} - \tfrac{654577}{1037312}y_{n+2} + \tfrac{51359}{32416}y_{n+\frac{5}{2}} + \tfrac{3465}{32416}h^2 f_{n+\frac{11}{4}} - \tfrac{41811}{1037312}h^2 f_{n+3} \\ y_{n+2} &= -\tfrac{273}{4961}y_n + \tfrac{3002}{3157}y_{n+1} - \tfrac{1376}{451}y_{n+\frac{5}{2}} + \tfrac{109568}{34727}y_{n+\frac{11}{4}} - \tfrac{360}{451}h^2 f_{n+2} - \tfrac{3}{41}h^2 f_{n+3} \end{aligned}\right\} \tag{14}$$

The equations (12) – (14) are the proposed 3-Step Block Hybrid Backward Differentiation Formulae (BHBDF) for solving second order ordinary differential equations.

2 Numerical Experiments

Problem. Variable Coefficient Non-Linear Type

$$\frac{d^2y(t)}{dt^2} = t\left(\frac{dy(t)}{dt}\right)^2 \qquad y(0) = 1,\ y'(0) = \frac{1}{2}$$

Exact Solution $y(t) = 1 + \frac{1}{2}\ln\left(\frac{2+t}{2-t}\right)$

t	**Exact**	**3SBHBDF**	**AbsoluteError**
0.0	1.0000000000000000000	1.0000000000000000000	$0.00000000000 \times 10^{0}$
0.1	1.0500417292784912682	1.0500417292741132458	$4.3780224000 \times 10^{-12}$
0.2	1.1003353477310755806	1.1003353477024359703	$2.8639610300 \times 10^{-11}$
0.3	1.1511404359364668053	1.1511404358437530280	$9.2713777300 \times 10^{-11}$
0.4	1.2027325540540821910	1.2027325538286751584	$2.2540703260 \times 10^{-10}$
0.5	1.2554128118829953416	1.2554128114150377147	$4.679576269 \times 10^{-10}$
0.6	1.3095196042031117155	1.3095196033199264267	$8.8318528880 \times 10^{-10}$
0.7	1.3654437542713961691	1.3654437526905396956	$1.5808564735 \times 10^{-9}$
0.8	1.4236489301936018069	1.4236489274514709661	$2.7421308408 \times 10^{-9}$
0.9	1.4847002785940517416	1.4847002739123865338	$4.6816652078 \times 10^{-9}$
1.0	1.5493061443340548457	1.5493061363158179051	$8.0182369406 \times 10^{-9}$

Table 1: Comparison of results for Problem 3 for $h = 0.01$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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