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Integrity of Variety of Inequalities Sketched on Time Scales

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Abstract: In this paper, we present extensions of some versions of the dynamic Rogers–Hölder inequality on time scales. Furthermore, we give an extension of integral inequality on time scales. To conclude this research article, we investigate some dynamic Jensen–type inequalities on time scales. Our approach unifies and extends some continuous inequalities and their corresponding discrete and quantum analogues.

Keywords: Time scales, Rogers–Hölder's inequality, integral inequality, Jensen-type inequality

1 Introduction

The following reverse dynamic Rogers–Hölder inequality on time scales is proved in [13].

Lemma 1.*Let $a,b \in \mathbb{T}$ and $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, $q>1$. If two positive functions $f,g:[a,b]\to\mathbb{R}$ are right-dense continuous (rd–continuous) and satisfying $0<m\leq\frac{f^p}{g^q}\leq M<\infty$ on the set $[a,b]_{\mathbb{T}}$, then we have the following inequality*

$$\left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)\Delta x\right)^{\frac{1}{q}}\leq\left(\frac{M}{m}\right)^{\frac{1}{pq}}\int_a^b f(x)g(x)\Delta x. \tag{1}$$

The following inequality is proved in [15].

Theorem 1.*Let $f,g:[a,b]\to\mathbb{R}$ be two positive functions satisfying $0<m\leq\frac{f(x)}{g(x)}\leq M<\infty$, $\forall x\in[a,b]$. Then we have the following inequality*

$$\frac{1}{M}\int_a^b f(x)g(x)dx\leq\frac{1}{(m+1)(M+1)}\int_a^b(f(x)+g(x))^2\,dx\leq\frac{1}{m}\int_a^b f(x)g(x)dx. \tag{2}$$

Recently, the following generalized Jensen–type inequality is obtained, see [7].

Theorem 2.*Let $\Phi:I\to\mathbb{R}$ be a continuous function on $I=[c,d]$, twice differentiable on (c,d), $c,d\in\mathbb{R}$, $x=(x_1,x_2,\dots,x_n)\in I^n$ $(n\geq 2)$. Let $w_i\geq 0$, $i=1,2,\dots,n$.*

(1°)*If there exists $m=\inf\limits_{x\in(c,d)}\Phi''(x)$, then*

$$\frac{\sum\limits_{i=1}^n w_i\Phi(x_i)}{\sum\limits_{i=1}^n w_i}-\Phi\left(\frac{\sum\limits_{i=1}^n w_ix_i}{\sum\limits_{i=1}^n w_i}\right)\geq\frac{m}{2}\left\{\frac{\sum\limits_{i=1}^n w_ix_i^2}{\sum\limits_{i=1}^n w_i}-\left(\frac{\sum\limits_{i=1}^n w_ix_i}{\sum\limits_{i=1}^n w_i}\right)^2\right\}. \tag{3}$$

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(2°)*If there exists* $M = \sup_{x\in(c,d)} \Phi''(x)$, *then*

$$\frac{\sum_{i=1}^{n} w_i\Phi(x_i)}{\sum_{i=1}^{n} w_i} - \Phi\left(\frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}\right) \leq \frac{M}{2}\left\{\frac{\sum_{i=1}^{n} w_i x_i^2}{\sum_{i=1}^{n} w_i} - \left(\frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}\right)^2\right\}. \tag{4}$$

We will unify and extend these results on time scales. The calculus of time scales was initiated by Stefan Hilger as given in [8]. A *time scale* is an arbitrary nonempty closed subset of the real numbers. The time scales calculus is studied as delta calculus, nabla calculus and diamond–α calculus. This hybrid theory is also widely applied on dynamic inequalities. Rogers–Hölder's inequality and its various extensions, generalizations and refinements play a very important role in mathematical analysis. Jensen's inequality is of great interest in the theory of differential, difference and quantum equations and in other areas of mathematics.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a,b \in \mathbb{T}$ with $a < b$ and an interval $[a,b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

2 Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adopted from monographs [4,5].
For $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0,+\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the *forward graininess function*. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0,+\infty)$ such that $\nu(t) := t - \rho(t)$ is called the *backward graininess function*. If $\sigma(t) > t$, we say that t is *right–scattered*, while if $\rho(t) < t$, we say that t is *left–scattered*. Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called *right–dense*, and if $t > \inf\mathbb{T}$ and $\rho(t) = t$, then t is called *left–dense*. If $\mathbb{T}$ has a left–scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^{Δ} is defined as follows:
Let $t \in \mathbb{T}^k$. If there exists $f^{\Delta}(t) \in \mathbb{R}$ such that for all $\varepsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be *delta differentiable* at t, and $f^{\Delta}(t)$ is called the *delta derivative* of f at t.
A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *right–dense continuous* (*rd–continuous*), if it is continuous at each right–dense point and there exists a finite left–sided limit at every left–dense point. The set of all rd–continuous functions is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.
The next definition is given in [4,5].

Definition 1. *A function* $F : \mathbb{T} \to \mathbb{R}$ *is called a delta antiderivative of* $f : \mathbb{T} \to \mathbb{R}$*, provided that* $F^{\Delta}(t) = f(t)$ *holds for all* $t \in \mathbb{T}^k$*. Then the delta integral of* f *is defined by*

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [3,4,5].
If $\mathbb{T}$ has a right–scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Further, $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$. A function $f : \mathbb{T}_k \to \mathbb{R}$ is called *nabla differentiable* at $t \in \mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t) \in \mathbb{R}$ such that given any $\varepsilon > 0$, there is a neighborhood V of t, such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *left–dense continuous* or *ld–continuous*, provided it is continuous at all left–dense points in $\mathbb{T}$ and its right–sided limits exist (finite) at all right–dense points in $\mathbb{T}$. The set of all ld–continuous functions is denoted by $C_{ld}(\mathbb{T},\mathbb{R})$.
The next definition is given in [3,4,5].

Definition 2. *A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by*

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Now we present short introduction of diamond–α derivative as given in [2,14].

Definition 3. *Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}$, the diamond–α dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by*

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad 0 \le \alpha \le 1.$$

Thus f is diamond–α differentiable if and only if f is Δ and ∇ differentiable.

The diamond–α derivative reduces to the standard Δ–derivative for $\alpha = 1$, or the standard ∇–derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0,1)$.

Definition 4([14]). *Let $a,t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond–$\alpha$ integral from a to t of h is defined by*

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s, \quad 0 \le \alpha \le 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 3([14]). *Let $a,b,t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$–integrable functions on $[a,b]_{\mathbb{T}}$. Then*

(i) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$;
(ii) $\int_a^t c f(s) \diamond_\alpha s = c \int_a^t f(s) \diamond_\alpha s$;
(iii) $\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s$;
(iv) $\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s$;
(v) $\int_a^a f(s) \diamond_\alpha s = 0$.

Theorem 4([2]). *Let $a,b \in \mathbb{T}$ and $c,d \in \mathbb{R}$. Suppose that $g \in C([a,b]_{\mathbb{T}},(c,d))$ and $w \in C([a,b]_{\mathbb{T}},\mathbb{R})$ with $\int_a^b |w(s)| \diamond_\alpha s > 0$. If $\Phi \in C((c,d),\mathbb{R})$ is convex, then generalized Jensen's inequality is*

$$\Phi\left(\frac{\int_a^b |w(s)| g(s) \diamond_\alpha s}{\int_a^b |w(s)| \diamond_\alpha s}\right) \le \frac{\int_a^b |w(s)| \Phi(g(s)) \diamond_\alpha s}{\int_a^b |w(s)| \diamond_\alpha s}. \tag{5}$$

If Φ is strictly convex, then the inequality $\le$ can be replaced by $<$.

Definition 5([6]). *A function $f : \mathbb{T} \to \mathbb{R}$ is called convex on $I_{\mathbb{T}} = I \cap \mathbb{T}$, where I is an interval of $\mathbb{R}$ (open or closed), if*

$$f(\lambda t + (1-\lambda)s) \le \lambda f(t) + (1-\lambda) f(s), \tag{6}$$

for all $t,s \in I_{\mathbb{T}}$ and all $\lambda \in [0,1]$ such that $\lambda t + (1-\lambda)s \in I_{\mathbb{T}}$.
The function f is strictly convex on $I_{\mathbb{T}}$ if the inequality (6) is strict for distinct $t,s \in I_{\mathbb{T}}$ and $\lambda \in (0,1)$.
The function f is concave (respectively, strictly concave) on $I_{\mathbb{T}}$, if $-f$ is convex (respectively, strictly convex).

3 Main Results

In order to present our main results, first we give a simple proof for an extension of dynamic Rogers–Hölder's inequality on time scales.

Theorem 5. *Let $w, f, g, h \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\diamond_\alpha$–integrable functions. Assume further that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=0$ for $p>0$, $q>0$, $r<0$ and $0<M^r \leq \frac{|g(x)|^q}{|f(x)|^p} \leq m^r < \infty$ on the set $[a,b]_{\mathbb{T}}$ satisfying $|f(x)g(x)h(x)|=c$, where c is a positive real number. Then the following inequality holds true.*

$$\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}} \left(\int_a^b |w(x)||h(x)|^r \diamond_\alpha x\right)^{\frac{1}{r}} \leq c\left(\frac{M}{m}\right)^{\frac{r^2}{pq}}. \tag{7}$$

Proof. It is given that

$$|f(x)|^p \leq M^{-r}|g(x)|^q \Rightarrow |f(x)|^{\frac{-rp}{q}} \leq M^{\frac{r^2}{q}}|g(x)|^{-r}.$$

Multiplying both sides of the last inequality by $|f(x)|^{-r}$, we get

$$|f(x)|^p \leq M^{\frac{r^2}{q}}|f(x)|^{-r}|g(x)|^{-r}.$$

Now, multiplying both sides of the last inequality by $|w(x)|$ and integrating it over x from a to b, we obtain

$$\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \leq M^{\frac{r^2}{pq}} \left(\int_a^b |w(x)||f(x)|^{-r}|g(x)|^{-r} \diamond_\alpha x\right)^{\frac{1}{p}}. \tag{8}$$

We also note that

$$|g(x)|^q \leq m^r|f(x)|^p \Rightarrow |g(x)|^{\frac{-rq}{p}} \leq m^{\frac{-r^2}{p}}|f(x)|^{-r}.$$

Multiplying both sides of the last inequality by $|g(x)|^{-r}$, we get

$$|g(x)|^q \leq m^{\frac{-r^2}{p}}|f(x)|^{-r}|g(x)|^{-r}.$$

Now, multiplying both sides of the last inequality by $|w(x)|$ and integrating it over x from a to b, we obtain

$$\left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}} \leq m^{\frac{-r^2}{pq}} \left(\int_a^b |w(x)||f(x)|^{-r}|g(x)|^{-r} \diamond_\alpha x\right)^{\frac{1}{q}}. \tag{9}$$

Multiplying (8) with (9), we get

$$\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{r^2}{pq}} \left(\int_a^b |w(x)||f(x)|^{-r}|g(x)|^{-r} \diamond_\alpha x\right)^{\frac{1}{p}+\frac{1}{q}}. \tag{10}$$

Using the condition $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=0$ and also applying $|f(x)g(x)h(x)|=c$, inequality (10) becomes

$$\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{r^2}{pq}} \left(\int_a^b |w(x)|c^{-r}|h(x)|^r \diamond_\alpha x\right)^{-\frac{1}{r}}. \tag{11}$$

The inequality (11) directly yields (7).

Next, we give another extension of dynamic Rogers–Hölder's inequality via time scales.

Theorem 6. *Let $w, f, g, h \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\diamond_\alpha$–integrable functions. Assume further that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=0$ for $p<0$, $q<0$, $r>0$ and $0<m^r \leq \frac{|g(x)|^q}{|f(x)|^p} \leq M^r < \infty$ on the set $[a,b]_{\mathbb{T}}$ satisfying $|f(x)g(x)h(x)|=c$, where c is a positive real number. Then the following inequality holds true.*

$$\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}} \left(\int_a^b |w(x)||h(x)|^r \diamond_\alpha x\right)^{\frac{1}{r}} \geq c\left(\frac{m}{M}\right)^{\frac{r^2}{pq}}. \tag{12}$$

Proof. Similar to the proof of Theorem 5.

Remark. Let $\alpha=1$, $w\equiv 1$, $r=-1$, $c=1$ and $f(x), g(x) \in (0,+\infty)$ on the set $[a,b]_{\mathbb{T}}$. Then inequality (7) reduces to (1).

Example 1. If $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k$, $g(k) = y_k$, $h(k) = z_k$ for $k \in \{1,2,...,n\}$, $n \in \mathbb{N}$, where x_k, y_k, z_k are the sets of positive values and $c = \left(\frac{m}{M}\right)^{\frac{r^2}{pq}}$, then discrete version of (7) takes the form

$$\left(\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} y_k^q\right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} z_k^r\right)^{\frac{1}{r}} \leq 1 \tag{13}$$

and discrete version of inequality (12) for $c = \left(\frac{M}{m}\right)^{\frac{r^2}{pq}}$ becomes

$$\left(\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} y_k^q\right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} z_k^r\right)^{\frac{1}{r}} \geq 1. \tag{14}$$

Inequalities (13) and (14) are given in [1, page 147] and such inequalities on time scales by using diamond–α integral are recently proved in [9]. A functional generalization of inequalities (13) and (14) may be found in [10].

Theorem 7. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\diamond_\alpha$–integrable functions. Assume further that $0 < m \leq \frac{|f(x)|}{|g(x)|} \leq M < \infty$ on the set $[a,b]_{\mathbb{T}}$. Then the following inequality holds true for $c > 0$:*

$$\frac{1}{M}\int_a^b |w(x)||f(x)g(x)|\diamond_\alpha x \leq \frac{1}{(m+c)(M+c)}\int_a^b |w(x)|\left(|f(x)|+c|g(x)|\right)^2 \diamond_\alpha x \leq \frac{1}{m}\int_a^b |w(x)||f(x)g(x)|\diamond_\alpha x. \tag{15}$$

Proof. From $0 < m \leq \frac{|f(x)|}{|g(x)|} \leq M$, it holds that

$$(m+c)|g(x)| \leq |f(x)| + c|g(x)| \leq (M+c)|g(x)|, \tag{16}$$

and

$$\frac{1}{M} \leq \frac{|g(x)|}{|f(x)|} \leq \frac{1}{m}.$$

By adding $\frac{1}{c}$ to both sides of the last inequality, we have that

$$\frac{M+c}{Mc} \leq \frac{|f(x)|+c|g(x)|}{c|f(x)|} \leq \frac{m+c}{mc}.$$

Multiplying $c|f(x)|$ to both sides, we get

$$\frac{M+c}{M}|f(x)| \leq |f(x)| + c|g(x)| \leq \frac{m+c}{m}|f(x)|. \tag{17}$$

Multiplying inequality (16) by inequality (17), then by $|w(x)|$ and integrating the results over x from a to b, we get

$$\begin{aligned}\frac{(m+c)(M+c)}{M}\int_a^b |w(x)||f(x)g(x)|\diamond_\alpha x \leq \int_a^b |w(x)|\left(|f(x)|+c|g(x)|\right)^2 \diamond_\alpha x \\ \leq \frac{(m+c)(M+c)}{m}\int_a^b |w(x)||f(x)g(x)|\diamond_\alpha x.\end{aligned} \tag{18}$$

The inequality (18) directly yields (15).

Remark. Let $\mathbb{T} = \mathbb{R}$, $c = 1$, $w \equiv 1$ and $f(x), g(x) \in (0, +\infty)$ on the set $[a,b]_{\mathbb{R}}$. Then inequality (15) reduces to the inequality (2).

Next, we present an extension of dynamic Jensen's inequality and its reverse on time scales.

Theorem 8. *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C([a,b]_{\mathbb{T}}, (c,d))$ and $w \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b |w(s)|\diamond_\alpha s > 0$. Let $\Phi \in C^2((c,d), \mathbb{R})$ be twice differentiable.*

(1°)*If there exists* $m = \inf_{s\in(c,d)} \Phi''(s)$, *then*

$$\frac{\int_a^b |w(s)|\Phi(g(s))\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} - \Phi\left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right) \geq \frac{m}{2}\left\{\frac{\int_a^b |w(s)|(g(s))^2\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} - \left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right)^2\right\}. \tag{19}$$

(2°)*If there exists* $M = \sup_{s\in(c,d)} \Phi''(s)$, *then*

$$\frac{\int_a^b |w(s)|\Phi(g(s))\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} - \Phi\left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right) \leq \frac{M}{2}\left\{\frac{\int_a^b |w(s)|(g(s))^2\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} - \left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right)^2\right\}. \tag{20}$$

*Proof.*Case (1°). We consider here

$$\Psi(s) = \Phi(s) - \frac{m}{2}s^2, \quad \forall s \in (c,d).$$

Obviously Ψ is a convex function and from Jensen's inequality (5), we obtain

$$\Phi\left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right) - \frac{m}{2}\left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right)^2 \leq \frac{\int_a^b |w(s)|\Phi(g(s))\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} - \frac{m}{2}\frac{\int_a^b |w(s)|g^2(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}. \tag{21}$$

Now, (21) takes the form

$$\frac{\int_a^b |w(s)|\Phi(g(s))\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} - \Phi\left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right) \geq \frac{m}{2}\left\{\frac{\int_a^b |w(s)|g^2(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} - \left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right)^2\right\}. \tag{22}$$

Case (2°). Let us consider that

$$\Psi(s) = \Phi(s) - \frac{M}{2}s^2, \quad \forall s \in (c,d).$$

Obviously Ψ is a concave function and from reverse of Jensen's inequality (5), we obtain our claim.

*Remark.*Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w(i) = w_i$ and $g(i) = x_i$ for $i = 1,2,\ldots,n$. Then inequality (19) reduces to the inequality (3) and inequality (20) reduces to the inequality (4).

Corollary 1.*Let* $a,b \in \mathbb{T}$. *Suppose that* $g \in C([a,b]_{\mathbb{T}},(0,+\infty))$ *and* $w \in C([a,b]_{\mathbb{T}},\mathbb{R})$ *with* $\int_a^b |w(s)|\diamond_\alpha s > 0$.

(1°)*If there exists* $m = \inf_{s\in(0,+\infty)} \frac{1}{s^2}$, *then*

$$\ln\left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right) - \frac{\int_a^b |w(s)|\ln(g(s))\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} \geq \frac{m}{2}\left\{\frac{\int_a^b |w(s)|(g(s))^2\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} - \left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right)^2\right\}. \tag{23}$$

(2°)*If there exists* $M = \sup_{s\in(0,+\infty)} \frac{1}{s^2}$, *then*

$$\ln\left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right) - \frac{\int_a^b |w(s)|\ln(g(s))\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} \leq \frac{M}{2}\left\{\frac{\int_a^b |w(s)|(g(s))^2\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s} - \left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right)^2\right\}. \tag{24}$$

*Proof.*Apply Theorem 8 for $\Phi(s) = -\ln(s)$ with domain $(0,+\infty)$.

Corollary 2.*Let* $a,b \in \mathbb{T}$. *Suppose that* $g \in C([a,b]_{\mathbb{T}},(-\infty,+\infty))$ *and* $w \in C([a,b]_{\mathbb{T}},\mathbb{R})$ *with* $\int_a^b |w(s)|\diamond_\alpha s > 0$.

(1°)*If there exists* $m = \inf\limits_{s\in(-\infty,+\infty)} \exp(s)$, *then*

$$\frac{\int_a^b |w(s)| \exp(g(s)) \diamond_\alpha s}{\int_a^b |w(s)| \diamond_\alpha s} - \exp\left(\frac{\int_a^b |w(s)| g(s) \diamond_\alpha s}{\int_a^b |w(s)| \diamond_\alpha s}\right) \geq \frac{m}{2}\left\{\frac{\int_a^b |w(s)| (g(s))^2 \diamond_\alpha s}{\int_a^b |w(s)| \diamond_\alpha s} - \left(\frac{\int_a^b |w(s)| g(s) \diamond_\alpha s}{\int_a^b |w(s)| \diamond_\alpha s}\right)^2\right\}. \quad (25)$$

(2°)*If there exists* $M = \sup\limits_{s\in(-\infty,+\infty)} \exp(s)$, *then*

$$\frac{\int_a^b |w(s)| \exp(g(s)) \diamond_\alpha s}{\int_a^b |w(s)| \diamond_\alpha s} - \exp\left(\frac{\int_a^b |w(s)| g(s) \diamond_\alpha s}{\int_a^b |w(s)| \diamond_\alpha s}\right) \leq \frac{M}{2}\left\{\frac{\int_a^b |w(s)| (g(s))^2 \diamond_\alpha s}{\int_a^b |w(s)| \diamond_\alpha s} - \left(\frac{\int_a^b |w(s)| g(s) \diamond_\alpha s}{\int_a^b |w(s)| \diamond_\alpha s}\right)^2\right\}. \quad (26)$$

Proof. Apply Theorem 8 for $\Phi(s) = \exp(s) = e^s$ with domain $(-\infty, +\infty)$.

Corollary 3. *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C([a,b]_{\mathbb{T}}, (c,d))$. Let $\Phi \in C^2((c,d), \mathbb{R})$ be twice differentiable.*

(1°)*If there exists* $m = \inf\limits_{s\in(c,d)} \Phi''(s)$, *then*

$$\frac{\int_a^b \Phi(g(s)) \diamond_\alpha s}{b-a} - \Phi\left(\frac{\int_a^b g(s) \diamond_\alpha s}{b-a}\right) \geq \frac{m}{2}\left\{\frac{\int_a^b (g(s))^2 \diamond_\alpha s}{b-a} - \left(\frac{\int_a^b g(s) \diamond_\alpha s}{b-a}\right)^2\right\}. \quad (27)$$

(2°)*If there exists* $M = \sup\limits_{s\in(c,d)} \Phi''(s)$, *then*

$$\frac{\int_a^b \Phi(g(s)) \diamond_\alpha s}{b-a} - \Phi\left(\frac{\int_a^b g(s) \diamond_\alpha s}{b-a}\right) \leq \frac{M}{2}\left\{\frac{\int_a^b (g(s))^2 \diamond_\alpha s}{b-a} - \left(\frac{\int_a^b g(s) \diamond_\alpha s}{b-a}\right)^2\right\}. \quad (28)$$

Proof. Apply Theorem 8 for $w \equiv 1$.

Example 2. Let $\mathbb{T} = \mathbb{R}_0^+$, $a = 0$, $b = 1$ and $w \equiv 1$. Then (19) and (20), respectively, take the forms

$$(1^\circ) \quad \int_0^1 \Phi(g(s))\,ds - \Phi\left(\int_0^1 g(s)ds\right) \geq \frac{m}{2}\left\{\int_0^1 (g(s))^2\,ds - \left(\int_0^1 g(s)ds\right)^2\right\} \quad (29)$$

and

$$(2^\circ) \quad \int_0^1 \Phi(g(s))\,ds - \Phi\left(\int_0^1 g(s)ds\right) \leq \frac{M}{2}\left\{\int_0^1 (g(s))^2\,ds - \left(\int_0^1 g(s)ds\right)^2\right\}. \quad (30)$$

The inequalities (29) and (30) may be found in [7].

Remark. If we set $\alpha = 1$, then we get delta versions and if we set $\alpha = 0$, then we get nabla versions of diamond–α integral operator inequalities presented in this article.
Also, if we set $\mathbb{T} = \mathbb{Z}$, then we get discrete versions and if we set $\mathbb{T} = \mathbb{R}$, then we get continuous versions of diamond–α integral operator inequalities presented in this article.
We get quantum inequalities by setting $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$ of diamond–α integral operator inequalities presented in this article by using the formula

$$\int_{q^m}^{q^n} f(x) \diamond_\alpha x = (q-1)\sum_{i=m}^{n-1} q^i[\alpha f(q^i) + (1-\alpha) f(q^{i+1})], \quad (31)$$

for $m < n$, $m, n \in \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of nonnegative integers.

4 Conclusion

The theory of time scales is applied to combine results in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. Dynamic inequalities on time scales may be explored by using functional generalization, n–tuple diamond–alpha integral operator, Specht's ratio, Kantorovich's ratio and fractional calculus of time scales. By considering an axiomatic definition of fractional calculus on time scales, many results have been developed concerning the time scales Riemann–Liouville type fractional integrals, see [11,12]. In the future research, we will continue to investigate more results by applying such interesting techniques.

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