The examples of the Invariant Subspaces in R^3: The G-Orbits for Similarity Groups

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Abstract: Let $(G,*)$ be a group and X be a nonempty set and the group action $G:X$ be given. In this paper we studied the $G-$orbits, the invariant subspaces, in R^3 regarding as $G = S(3)$, all similarity transformations' group in three dimensional Euclidean space, and all subgroups of it.

Keywords: G-orbits, invariant subspaces, similarity groups

1 Introduction

Let $(G,*)$ be a group and X be a given nonempty set. For any element x, the set of images $g * x$ under the all $g \in G$ is calle the $G-$orbit of the element x. The smallest $G-$invariant subset containing element x is the $G-$orbit of x. The $G-$invariant subset also covers the $G-$orbit Gx if it covers element x.The problem of finding invariant subspaces is widely used in mathematics. Especially it is a concept used in operator theory in Hilbert spaces and used for solutions of nonlinear differential equations and for complete continuous operators. [2],[3], [4], [5].

The first group action studied was the action of the Galois group on the roots of a polynomial. However, there are a lot of examples and applications of group actions in many areas of mathematics as well as chemistry and physics. In general, an orbit may be of any dimension, up to the dimension of the Lie group. If the Lie group G is compact, then its orbits are submanifolds. [1].

It is seen that orbit spaces has been studied in many areas in invariant theory. [20], [21], [22], [23], [24], [25], [26], [27], [28], [29]. In 1897 E. Study gave the full invariant system of the points for the group $O(n)$. Then Hermann Weyl developed this subject later him [6]. In 1988, Djavvat Khadjiev, R. Aripov solved this problem for the whole Euclidean movement group$E(n)$ [7]. Then Weyl representations have been applied to a wide variety of areas. These applications have been expanded in many fields ranging from the classification of bacteria to the problems of perception. For example, E. Cassier [8] gave the invariants of the transformation group as perceptual characterizations. Later Hoffman [9], [10] developed the theory of perception based on Lie transformation groups and Lie algebras. They were followed by Chan & Chan [11], [12] and Leyton [13]. Other studies on invariant parameterization and curves theory can be exemplified [14], [15], [16], [17], [18], [19], [30], [31], [32]. In this paper we studied the $G-$orbits, the invariant subspaces, in R^3 regarding as $G = S(3)$ and all subgroups of it.

2 The Group Actions

Definition 1.*Let $(G,*)$ be a group and X be a nonempty set. Let the transformation $\varphi : G \times X \to X$ be given. If following conditions;*

1.$\varphi(g_1,\varphi(g_2,x)) = \varphi(g_1 * g_2,x) \forall g_1,g_2 \in G$ and $\forall x \in X$,
2.$\varphi(e, x) = x$, for $\forall x \in X$ where $e \in G$ is identity element.

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satisfies then the transformation φ is called Group action of the group G on X. This action is denoted by $G : X$ and the image $\varphi(g, x)$ is stated as gx briefly.

Example 1. Let G be chosen as the linear homotethies' sets in three dimensional Euclidean spaces as follows

$$G = LH(3) = \left\{ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} : \lambda \in R^+ \right\}.$$

This set is a group with binary operation of the product of the matrices. The action of this group on E^3. Let $G : E^3$ be defined by $\varphi : G \times E^3 \to E^3$ such that $(g,x) \xrightarrow{\varphi} \varphi(g, x) = gx$ where $g = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. It is easily shown

that the action of $\varphi : G \times E^3 \to E^3$ is really the group action:

For $g_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$, $g_2 = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$, and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in E^3.$

$$
\begin{aligned}
1. \varphi(g_1, \varphi(g_2,x)) &= \varphi\left(\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \\
&= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} . \begin{bmatrix} \lambda_2 x_1 \\ \lambda_2 x_2 \\ \lambda_2 x_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \lambda_2 x_1 \\ \lambda_1 \lambda_2 x_2 \\ \lambda_1 \lambda_2 x_3 \end{bmatrix} \\
&= \varphi\left(\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} . \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \\
&= \varphi(g_1 g_2, x).
\end{aligned}
$$

$$
2. \varphi(e, x) = \varphi\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x, \quad \text{for every } x \in E^3.
$$

When any $g \in G$ be chosen and be fixed the transformation $\varphi : G \times X \to X$; $(g,x) \xrightarrow{\varphi} \varphi(g, x) = gx$ is reduced to the transformation $\varphi(g, .) : X \to X$; $x \xrightarrow{\varphi(g, .)} \varphi(g, x) = gx.$

Definition 2. Let G be a group and the group actions of G on the sets X_1 and X_2 such that $\varphi_1 = G : X_1$ and $\varphi_2 = G : X_2$ be given. If there exists a one-to-one and onto transformation $F : X_1 \longrightarrow X_2$ such that the diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{F} & X_2 \\
\varphi_1(g, .) \downarrow & & \downarrow \varphi_2(g, .) \\
X_1 & \xrightarrow{F} & X_2
\end{array}
$$

is commutative or such that $F \circ \varphi_1(g, .) = \varphi_2(g, .) \circ F$ is satisfied, then the actions $\varphi_1 = G : X_1$ and $\varphi_2 = G : X_2$ are called equivalent group actions.

Example 2.   Let $G = LH(3)$ be choosen and $X_1 = \{(x,y,z) : z > 0\} \subset R^3$ and $X_2 = \{(x,y,z) : z < 0\} \subset R^3$ be given. Two group actions of the group G are

$\varphi_1 = G : X_1$ such that

$$
\begin{aligned}
\varphi_1(g, (x,y,z)) &= \varphi_1 \left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, (x,y,z) \right) \\
&= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}
\end{aligned}
$$

Example 3. and $\varphi_2 = G : X_2$ such that

$$
\varphi_2(g, (x,y,z)) = \varphi_2 \left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, (x,y,z) \right)
$$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

be considered. In this case there exists a one-to-one and onto transformation $F : X_1 \longrightarrow X_2$ such that $F(x,y,z) = (x,y,-z)$ can be considered. Hence the diagram;

$$\begin{array}{ccc} X_1 & \xrightarrow{F} & X_2 \\ \varphi_1(g,\ .)\downarrow & & \downarrow \varphi_2(g,\ .) \\ X_1 & \xrightarrow{F} & X_2 \end{array}$$

is commutative. So these group actions are equivalent.

3   The $G-$orbits

Definition 3.*Let G be a group and the group action $\varphi = G : X$ and the subset $H \subset X$ be given. the subset H is called "G−**invariant subset**" if $\varphi(g,h) \in H$ for every $h \in H$ and for every $g \in G$.*

Considering $H = \{x_0\} \subset X$ in this definition, i.e. the point x_0 is called "G−invariant point" if $gx_0 = x_0$ for every $g \in G$.

Definition 4.*Let G be a group and the group action $\varphi = G : X$ be given. For any element $x \in X$ the set $Gx = \{gx : g \in G\}$ is called G−orbit of the element x.*

Proposition 1.*Let G be a group and the group action $\varphi = G : X$ be given. The G−orbit of the element x is G−invariant subset of X.*

*Proof.*To prove that $Gx \subset X$ is a $G-$invariant subset, we must show $gy \in Gx$ for every $y \in Gx$ and for every $g \in G$. Let $y \in Gx$ be given. In this case, there exists $g_1 \in G$ such that $y = g_1 x$ satisfies. So $gy = g(g_1 x)$ can be written. From defining group action it is true that $g(g_1 x) = (gg_1)x$. From that G is a group , it can be seen easily $gy = g^* x$ by writing g^* instead of gg_1.

Proposition 2.*Let G be a group and the group action $\varphi = G : X$ be given. There is no any G−invariant subset of Gx different from itself.*

*Proof.*Let $H \subset Gx$ be a $G-$invariant subset of Gx. we must prove that $H = Gx$. Since $e \in G$ is the identity element, $x \in Gx$. Let $y \in H$ be given. In this case, there exists $g_1 \in G$ such that $y = g_1 x \in H$ is satisfies. it is seen that $gy = g(g_1 x) \in H$ for every $g \in G$, since H is a $G-$invariant subset. So it is also true for $g = g_1^{-1}$. *It means that* $gy = g_1^{-1}(g_1 x) = x \in H$ is obtained. As a result of this $Gx \subset H$ satisfies since $x \in H$ is obtained when $x \in Gx$. So $H = Gx$ satisfies.

Corollary 1.*Let G be a group and the group action $\varphi = G : X$ be given. Then $Gx = Ga$ for any $a \in Gx$.*

*Proof.*It can be easily proved.

Proposition 3.*Let G be a group and the group action $\varphi = G : X$ be given. Then $Gx = Gy$ for different elements x and y of X if $Gx \cap Gy \neq \varnothing$. with other statement, $Gx \cap Gy \neq \varnothing$ if $Gx \neq Gy$ for different elements x and y of X.*

*Proof.*Let $Gx \cap Gy \neq \varnothing$ be supposed. In this case, there exists $a \in Gx$ and $a \in Gy$. From previous corollary $Ga = Gx$ and $Ga = Gy$ are concluded . So $Gx = Gy$ is obtained.

Definition 5.*Let G be a group and the group action $\varphi = G : X$ be given. For $x, y \in X$, if there exists $g \in G$ such that $y = gx$ satisfies, then the points x and y are called G− equivalent points and denoted by $x \overset{G}{\sim} y$.*

Proposition 4.*The definition of G−equivalence of points is an equivalence relationship.*

As a result of the definition of $G-$equivalence of points, the equivalence class of an element x according to the relationship $\overset{G}{\sim}$ is $G-$orbit Gx.

4 The Similarity Group $G = S(3)$ and its all subgroups in R^3

1- **Orthogonal Transformations' Group** $O(3)$ This group is a group of all rotations and reflections. Orthogonal Transformations' group is the same of the linear isometries' group. For any $g \in O(3)$, the determinant of g is equal ± 1. The rotations depend on the angles between rotating axis $\{e_1^*, e_2^*, e_3^*\}$ and fixed axis $\{e_1, e_2, e_3\}$ of the frames $\{e_1, e_2, e_3\}$. The angle θ_{ij}, is an angle between the axis $\{e_i^*, e_j\}$ So this group can be stated as

$$O(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = gx; \ g^T = g^{-1} \right\} = \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_{11} & \cos\theta_{21} & \cos\theta_{31} \\ \cos\theta_{12} & \cos\theta_{22} & \cos\theta_{32} \\ \cos\theta_{13} & \cos\theta_{23} & \cos\theta_{33} \end{bmatrix} : \theta_{ij} \in R \right\}.$$

2- **Special Orthogonal Transformations' Group** $SO(3)$

This group is a group of all rotations. For any $g \in SO(3)$, the determinant of g is equal $+1$. The rotations depend on the angles between rotating axis $\{e_1^*, e_2^*, e_3^*\}$ and fixed axis $\{e_1, e_2, e_3\}$ of the frames $\{e_1, e_2, e_3\}$. The angle θ_{ij}, is an angle between the axis $\{e_i^*, e_j\}$ So this group can be stated as

$$SO(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = gx; \ g^T = g^{-1}; \det g = +1 \right\} = \left\{ \begin{bmatrix} \cos\theta_{11} & \cos\theta_{21} & \cos\theta_{31} \\ \cos\theta_{12} & \cos\theta_{22} & \cos\theta_{32} \\ \cos\theta_{13} & \cos\theta_{23} & \cos\theta_{33} \end{bmatrix} : \theta_{ij} \in R \right\}.$$

3- **Translations' Group** $Tr(3)$

This group is a group of all translations.
$$Tr(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = x + b; \ b \in R^3 \right\}.$$

4- **Euclid Transformations' Group** $E(3)$

This group is a group of all translations, rotations, rotations and translations, reflections, reflections and translations. This group is the same of all isometries' group. Euclid transformations can be stated as a compositition of a translation and an orthogonal transformation. So this group can be stated as
$$E(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = gx + b; \ g \in O(3); \ b \in R^3 \right\}.$$

5- **Special Euclid Tranformations' Group** $SE(3)$

This group is a group of all translations, rotations, rotations and translations. Special Euclid transformations can be stated as a compositition of a translation and a special orthogonal transformation. So this group can be stated as
$$SE(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = gx + b; \ g \in SO(3); \ b \in R^3 \right\}.$$

6- **Linear Homotethies' Group** $LH(3)$

This group is a group of all central dilations or radyal transformations. Linear homotethies are the transformations that its center of homotethy is the origin. So this group can be stated as

$$LH(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = \lambda x; \ \lambda \in R^+ \right\}.$$

7- **Homotethies' Group** $H(3)$

The hometethy transformation can be defined as $f(x) = a + \lambda(x - a)$. where a is called homotethy center of f. This transformation can be stated as $f(x) = \lambda x + b$ for $\lambda \neq 1$ where the center of homnotethy is $a = \frac{b}{1-\lambda}$ So this group can be stataed as
$$H(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = \lambda x + b; \ \lambda \in R^+; b \in R^3 \right\}.$$

8- **Linear Similarities' Group** $LS(3)$

Linear similarity transformations can be stated as a compositition of a linear homotethy transformation and an orthogonal transformation. So linear similarity transformations can be stated as
$$LS(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = \lambda gx; \ \lambda \in R^+; \ g \in O(3) \right\}.$$

9- **Special Linear Similarities' Group** $SLS(3)$

Special linear similarity transformations can be stated as a compositition of a linear homotethy transformation and a special orthogonal transformation. So linear similarity transformations can be stated as
$$SLS(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = \lambda gx; \ \lambda \in R^+; \ g \in SO(3) \right\}.$$

10- **Special Similarities' Group** $SS(3)$

Special similarity transformations can be stated as a composititition of a special linear similarity transformation and a translation. In other words a special similarity transformation can be stated as a composititition of a linear homotethy transformation, a special orthogonal transformation, and a translation. So special similarity transformations can be stated as

$$SS(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = \lambda gx + b; \ \lambda \in R^+; \ g \in SO(3); \ b \in R^3 \right\}.$$

11- **All Similarities' Group** $S(3)$

Similarity transformations can be stated as a composititition of a linear similarity transformation and a translation. In other words a similarity transformation can be stated as a composititition of a linear homotethy transformation, an orthogonal transformation, and a translation. So similarity transformations can be stated as

$$S(3) = \left\{ f : R^3 \longrightarrow R^3 : f(x) = \lambda gx + b; \ \lambda \in R^+; \ g \in O(3); \ b \in R^3 \right\}.$$

5 The $G-$orbits for $G = S(3)$ and its all subgroups

Now let us obtain the $G-$orbits of R^3 regarding as $G = S(3)$ and its all subgroups.

Theorem 1. *The $O(3)-$orbits of R^3 consist of origin, and the surfaces of sphere with radius r. ($r \in R^+$)*

Proof. For $x = 0$, the $G-$orbit is $G0 = \{g.0 : g \in O(3)\} = \{0\}$

Let $x = (x_1, x_2, x_3)$ be a nonzero vector of R^3. Then, the $G-$orbit is $Gx = \{gx : g \in O(3)\}$. For the orthogonal transformations f, it is true that $\|f(x)\| = \|x\|$. So $\|gx\| = \|x\|$ satisfies. Then Choosing $\|x\| = r$, the $G-$orbit of x is the set consisting of the vectors x such that $x_1^2 + x_2^2 + x_3^2 = r^2$.(see. Fig. 1)

Theorem 2. *The $SO(3)-$orbits of R^3 consist of origin, and the surfaces of sphere with radius r. ($r \in R^+$)*

Proof. It can be easily seen similarly previous theorem. (see. Fig. 1)

Theorem 3. *The $Tr(3)-$orbit of R^3 is space R^3 itself.*

Proof. Since any element x can be translated to any element of the space R^3 the $Tr(3)-$orbit of R^3 is space R^3 itself.

Theorem 4. *The $E(3)-$orbit of R^3 is space R^3 itself.*

Proof. Similarly Since any element x can be rotated and reflected and then translated to any element of the space R^3 the $E(3)-$orbit of R^3 is space R^3 itself.

Theorem 5. *The $SE(3)-$orbit of R^3 is space R^3 itself.*

Proof. Similarly Since any element x can be rotated and then translated to any element of the space R^3 the $SE(3)-$orbit of R^3 is space R^3 itself.

Theorem 6. *The $LH(3)-$orbits of R^3 consist of origin, and rays distributed from origin.*

Proof. If $x = 0$ then the $G-$orbit of 0 is $G0 = \{g.0 : g \in O(3)\} = \{0\}$

If $x \neq 0$ then the $G-$orbit of x is $Gx = \{\lambda.x : \lambda > 0\}$. So $LH(3)-$orbits of R^3 consist of origin, and rays distributed from origin. (See Fig. 2)

Theorem 7. *The $H(3)-$orbit of R^3 is space R^3 itself.*

Proof. Since any element x can be dilated and then translated to any element of the space R^3 the $H(3)-$orbit of R^3 is space R^3 itself.

Theorem 8. *The $LS(3)-$orbits of R^3 consist of two orbits: origin, other part of the space R^3 except of the origin.*

Proof. If $x = 0$ then the $G-$orbit of 0 is $G0 = \{\lambda.g.0 : \lambda > 0, g \in O(3)\} = \{0\}$

If $x \neq 0$ then the $G-$orbit of x is $Gx = \{\lambda.g.x : \lambda > 0, g \in O(3)\} = R^3 \backslash \{0\}$. So $LS(3)-$orbits of R^3 consist of origin, and other part of the space R^3 except of the origin.

Theorem 9. *The $SLS(3)-$orbits of R^3 also consist of two orbits: origin, other part of the space R^3 except of the origin.*

*Proof.*If $x = 0$ then the $G-$orbit of 0 is $G0 = \{\lambda.g.0 : \lambda > 0, g \in SO(3)\} = \{0\}$

If $x \neq 0$ then the $G-$orbit of x is $Gx = \{\lambda.g.x : \lambda > 0, g \in SO(3)\} = R^3 \setminus \{0\}$. So $SLS(3)-$orbits of R^3 consist of origin, and other part of the space R^3 except of the origin.

Theorem 10.*The $SS(3)-orbit$ of R^3 is space R^3 itself.*

*Proof.*Similarly Since any element x can be rotated and dilated and then translated to any element of the space R^3 the $E(3)-$orbit of R^3 is space R^3 itself.
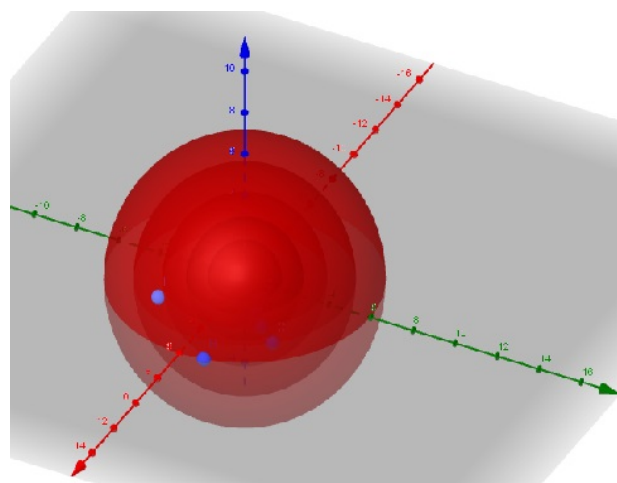
Theorem 11.*The $S(3)-orbit$ of R^3 is space R^3 itself.*

*Proof.*Similarly Since any element x can be rotated or reflected and dilated and then translated to any element of the space R^3 the $S(3)-$orbit of R^3 is space R^3 itself.

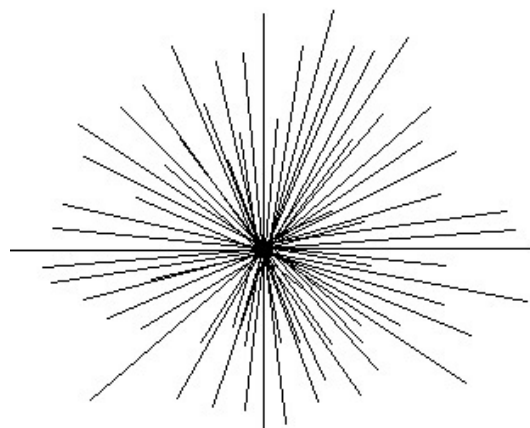**Fig. 1:** The $O(3)-$orbits and the $SO(3)-$orbits of R^3.

Fig. 2: The $LH(3)-$orbits of R^3.

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