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On Some Bullen Type Quantum Integral Inequalities

Musa ÇAKMAK

Hatay Mustafa Kemal University, Yayladağı Social Sciences Vocational School, 31500, Hatay, TURKEY

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Abstract: In this paper, Bullen type inequalities for quantum integral are studied and new integral identity including Bullen type identity for quantum integral is established using q-calculus. Second, some new integral inequalities including Bullen-type inequalities for quantum integral are generated using q-calculus. In addition, the same results were obtained with the existing studies in the literature.

Keywords: Bullen type inequality, fractional integrals, integral inequalities, $q-$calculus

1 Introduction

The Hermite-Hadamard inequality: Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $u, v \in I$ with $u < v$.

$$\varphi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \varphi(\varkappa)\, d\varkappa \leq \frac{\varphi(u)+\varphi(v)}{2}. \tag{H}$$

If φ is concave, the inequality of H is written in an inverse way. You can see [4,15,16] for details. The Bullen inequality:

$$\frac{1}{v-u}\int_u^v \varphi(\varkappa)\, d\varkappa \leq \frac{1}{2}\left[\frac{\varphi(u)+\varphi(v)}{2} + \varphi\left(\frac{u+v}{2}\right)\right], \tag{B}$$

provided that $\varphi : [u,v] \to \mathbb{R}$ is a convex function on $[u,v]$ (see [4,8,15,16,17,18,19,20]) for more details.

2 Quantum Fractional Derivative and Integral

Let $I = [u,v] \subset \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. We define $q-$derivative of a function $\varphi : I \longrightarrow \mathbb{R}$ at a point $x \in I$ on $[u,v]$ as follows[10].

Definition 1.*If $\varphi : I \longrightarrow \mathbb{R}$ is a continuous function and let $x \in I$. Then the following identities*

$$\begin{aligned} {}_uD_q(\varphi)(x) &= \frac{\varphi(x) - \varphi(qx + (1-q)u)}{(1-q)(x-u)},\ x \neq u, \\ {}_uD_q(\varphi)(u) &= \lim_{x \longrightarrow u} {}_uD_q(\varphi)(x), \end{aligned} \tag{1}$$

is called the $q-$derivative on I of a function φ at a point x. Also, if $u = 0$ in (1), then ${}_0D_q(\varphi)(u) = D_q\varphi$, where D_q is the $q-$derivative of the function $\varphi\ (x)$ defined by

$$D_q(\varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x}.$$

For more details, see [9].

* Corresponding author e-mail: musacakmak@mku.edu.tr

Definition 2.[12] *Assume* $\alpha \in \mathbb{R}$, *then we have*

$$ {}_{u}D_{q}(x-u)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q}\right)(x-u)^{\alpha-1}. \tag{2}$$

Definition 3.*Let* $\varphi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function. Then,* $q-$*integral on* I *is defined as*

$$\int_{u}^{x} \varphi(t)_{u} d_{q}t = (1-q)(x-u)\sum_{n=0}^{\infty} q^{n}\varphi(q^{n}x+(1-q^{n})u), \tag{3}$$

for $x \in I$. *If* $u = 0$ *in* (3), *then we have the classical* $q-$*integral* [9].

Moreover, $v \in (u,x)$ then the definite $q-$integral on I is defined by

$$\begin{aligned}\int_{v}^{x} \varphi(t)_{u} d_{q}t &= \int_{u}^{x} \varphi(t)_{u} d_{q}t - \int_{u}^{v} \varphi(t)_{u} d_{q}t \\ &= (1-q)(x-u)\sum_{n=0}^{\infty} q^{n}\varphi(q^{n}x+(1-q^{n})u) - (1-q)(v-u)\sum_{n=0}^{\infty} q^{n}\varphi(q^{n}v+(1-q^{n})u).\end{aligned}$$

Definition 4.[13] *For* $\alpha \in \mathbb{R} - \{-1\}$, *the following formula holds:*

$$\int_{u}^{x} (t-u)_{u}^{\alpha} d_{q}t = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-u)^{\alpha+1}.$$

In many studies in the literature [1,3,5,6,7,9,11,12,13,14], many integral inequalities have been obtained by using fractional integrals for differentiable convex functions. In this study, using the properties of $q-$calculus (fractional derivative and integral) and the following lemma, Bullen type integral inequality was obtained.

3 New Bullen-Type q-Fractional Integral Inequalities

Lemma 1.*Let* $f : I \subset \mathbb{R}^{+} \to \mathbb{R}$ *be a continuous function and* $0 < q < 1$. *If* ${}_{a}D_{q}f$ *is an integrable function on* I°*(the interior of* I*), then the following identity for* $q-$ *fractional integral holds:*

$$\int_{0}^{1} (1-2qt)\left[{}_{a}D_{q}(f)\left(\frac{a+b}{2}t+(1-t)a\right) + {}_{a}D_{q}(f)\left(bt+(1-t)\frac{a+b}{2}\right)\right] {}_{0}d_{q}t \tag{4}$$

$$= -\frac{2}{b-a}\left(f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right) + \frac{8}{(b-a)^{2}}\int_{a}^{b} f(x)\, {}_{a}d_{q}x \tag{5}$$

*Proof.*From (1) and (3), we have

$$\int_0^1 (1-2qt)\left({}_aD_q(f)\left(\frac{a+b}{2}t+(1-t)a\right)\right){}_0d_qt+\int_0^1(1-2qt)\left({}_aD_q(f)\left(bt+(1-t)\frac{a+b}{2}\right)\right){}_0d_qt$$
$$=\int_0^1 {}_aD_q f\left(\frac{a+b}{2}t+(1-t)a\right){}_0d_qt-2\int_0^1 qt\,{}_aD_q f\left(\frac{a+b}{2}t+(1-t)a\right){}_0d_qt$$
$$+\int_0^1 {}_aD_q f\left(bt+(1-t)\frac{a+b}{2}\right){}_0d_qt-2\int_0^1 qt\,{}_aD_q f\left(tb+(1-t)\frac{a+b}{2}\right){}_0d_qt$$
$$=2\int_0^1\frac{f\left(\frac{a+b}{2}t+(1-t)a\right)-f\left(\frac{a+b}{2}qt+(1-qt)a\right)}{(b-a)(1-q)t}{}_0d_qt-4\int_0^1 q\frac{f\left(\frac{a+b}{2}t+(1-t)a\right)-f\left(\frac{a+b}{2}qt+(1-qt)a\right)}{(b-a)(1-q)}{}_0d_qt$$
$$+2\int_0^1\frac{f\left(bt+(1-t)\frac{a+b}{2}\right)-f\left(bqt+(1-qt)\frac{a+b}{2}\right)}{(b-a)(1-q)t}{}_0d_qt-4\int_0^1 q\frac{f\left(bt+(1-t)\frac{a+b}{2}\right)-f\left(bqt+(1-qt)\frac{a+b}{2}\right)}{(b-a)(1-q)}{}_0d_qt$$
$$=2\sum_{n=0}^{\infty}\frac{f\left(\frac{a+b}{2}q^n+(1-q^n)a\right)}{(a-b)}-2\sum_{n=0}^{\infty}\frac{f\left(\frac{a+b}{2}q^{n+1}+(1-q^{n+1})a\right)}{(b-a)}-4\sum_{n=0}^{\infty}q^{n+1}\frac{f\left(\frac{a+b}{2}q^n+(1-q^n)a\right)}{(b-a)}$$
$$+4\sum_{n=0}^{\infty}q^{n+1}\frac{f\left(\frac{a+b}{2}q^{n+1}+(1-q^{n+1})a\right)}{(b-a)}+2\sum_{n=0}^{\infty}\frac{f\left(bq^n+(1-q^n)\frac{a+b}{2}\right)}{(b-a)}-2\sum_{n=0}^{\infty}\frac{f\left(bq^{n+1}+(1-q^{n+1})\frac{a+b}{2}\right)}{(b-a)}$$
$$-4\sum_{n=0}^{\infty}q^{n+1}\frac{f\left(bq^n+(1-q^n)\frac{a+b}{2}\right)}{(b-a)}+4\sum_{n=0}^{\infty}q^{n+1}\frac{f\left(bq^{n+1}+(1-q^{n+1})\frac{a+b}{2}\right)}{(b-a)}.$$

$$=2\sum_{n=0}^{\infty}\frac{f\left(\frac{a+b}{2}q^n+(1-q^n)a\right)}{(b-a)}-2\sum_{n=1}^{\infty}\frac{f\left(\frac{a+b}{2}q^n+(1-q^n)a\right)}{(b-a)}-4\sum_{n=0}^{\infty}q^{n+1}\frac{f\left(\frac{a+b}{2}q^n+(1-q^n)a\right)}{(b-a)}$$
$$+4\sum_{n=1}^{\infty}q^{n}\frac{f\left(\frac{a+b}{2}q^n+(1-q^n)a\right)}{(b-a)}+2\sum_{n=0}^{\infty}\frac{f\left(bq^n+(1-q^n)\frac{a+b}{2}\right)}{(b-a)}-2\sum_{n=1}^{\infty}\frac{f\left(bq^n+(1-q^n)\frac{a+b}{2}\right)}{(b-a)}$$
$$-4\sum_{n=0}^{\infty}q^{n+1}\frac{f\left(bq^n+(1-q^n)\frac{a+b}{2}\right)}{(b-a)}+4\sum_{n=1}^{\infty}q^{n}\frac{f\left(bq^n+(1-q^n)\frac{a+b}{2}\right)}{(b-a)}$$
$$=\frac{2f\left(\frac{a+b}{2}\right)}{(b-a)}-\frac{2f(a)}{(b-a)}-\frac{4f\left(\frac{a+b}{2}\right)}{(b-a)}+\frac{4}{(b-a)}\int_0^1 f\left(\frac{a+b}{2}t+(1-t)a\right){}_0d_qt+\frac{2f(b)}{(b-a)}-\frac{2f\left(\frac{a+b}{2}\right)}{(b-a)}-\frac{4f(b)}{(b-a)}$$
$$+\frac{4}{(b-a)}\int_0^1 f\left(bt+(1-t)\frac{a+b}{2}\right){}_0d_qt$$
$$=-\frac{2}{(b-a)}\left(f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right)+\frac{8}{(b-a)^2}\int_a^{\frac{a+b}{2}}f(x)\,{}_ad_qx+\frac{8}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(x)\,{}_ad_qx$$
$$=-\frac{2}{(b-a)}\left(f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right)+\frac{8}{(b-a)^2}\int_a^b f(x)\,{}_ad_qx\,.$$

Theorem 1.*Let $f:I\to R$ be a continuous function and $0<q<1$. If $|{}_aD_\alpha f\,|$ is a convex and integrable function on I°, then the following inequality holds:*

$$\left|-\frac{1}{4}\left(f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right)+\frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right|\le\left(\frac{b-a}{16(1+q)}\right)\left[\frac{1+2q^3}{(q^2+q+1)}\left|{}_aD_q(f)(a)\right|\right.$$
$$\left.+2q\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|+\frac{2q^2+2q-1}{(q^2+q+1)}\left|{}_aD_q(f)(b)\right|\right].$$

*Proof.*Since $|{}_aD_\alpha f\,|$ is a convex function on I°, by the using Lemma 1 and using the well known absolute value inequality, and we have

$$\begin{aligned}
&\left|-\frac{1}{4}\left(f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right)+\frac{1}{(b-a)}\int_a^b f(x)\,{}_ad_qx\right| \\
&=\left|\left(\frac{b-a}{8}\right)\int_0^1(1-2qt)\left[{}_aD_q(f)\left(\frac{a+b}{2}t+(1-t)a\right)\ +{}_aD_q(f)\left(bt+(1-t)\frac{a+b}{2}\right)\right]{}_0d_qt\right| \\
&\le\left(\frac{b-a}{8}\right)\int_0^1|1-2qt|\left|{}_aD_q(f)\left(\frac{a+b}{2}t+(1-t)a\right)\right|{}_0d_qt+\left(\frac{b-a}{8}\right)\int_0^1|1-2qt|\left|{}_aD_q(f)\left(bt+(1-t)\frac{a+b}{2}\right)\right|{}_0d_qt \\
&\le\left(\frac{b-a}{8}\right)\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|\int_0^1|1-2qt|t\,{}_0d_qt+\left(\frac{b-a}{8}\right)\left|{}_aD_q(f)(a)\right|\int_0^1|1-2qt|(1-t)\,{}_0d_qt \\
&+\left(\frac{b-a}{8}\right)\left|{}_aD_q(f)(b)\right|\int_0^1|1-2qt|t\,{}_0d_qt+\left(\frac{b-a}{8}\right)\left|{}_aD_q(f)\left((1-t)\frac{a+b}{2}\right)\right|\int_0^1|1-2qt|(1-t)\,{}_0d_qt \\
&=\left(\frac{b-a}{16}\right)\frac{2q^2+2q-1}{(1+q)(q^2+q+1)}\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|+\left(\frac{b-a}{16}\right)\frac{1+2q^3}{(1+q)(q^2+q+1)}\left|{}_aD_q(f)(a)\right| \\
&+\left(\frac{b-a}{16}\right)\frac{2q^2+2q-1}{(1+q)(q^2+q+1)}\left|{}_aD_q(f)(b)\right|+\left(\frac{b-a}{16}\right)\frac{1+2q^3}{(1+q)(q^2+q+1)}\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right| \\
&=\left(\frac{b-a}{16(1+q)}\right)\left[\frac{1+2q^3}{(q^2+q+1)}\left|{}_aD_q(f)(a)\right|\ +2q\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|+\frac{2q^2+2q-1}{(q^2+q+1)}\left|{}_aD_q(f)(b)\right|\right]
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^1|1-2qt|t\,{}_0d_qt=\int_0^{\frac{1}{2q}}(1-2qt)t\,{}_0d_qt+\int_{\frac{1}{2q}}^1(2qt-1)t\,{}_0d_qt \\
&=\int_0^{\frac{1}{2q}}(1-2qt)t\,{}_0d_qt+\int_0^1(2qt-1)t\,{}_0d_qt-\int_0^{\frac{1}{2q}}(2qt-1)t\,{}_0d_qt=\frac{1}{2}\frac{2q^2+2q-1}{(1+q)(q^2+q+1)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1|1-2qt|(1-t)\,{}_0d_qt=\int_0^{\frac{1}{2q}}(1-2qt)(1-t)\,{}_0d_qt+\int_{\frac{1}{2q}}^1(2qt-1)(1-t)\,{}_0d_qt \\
&=\int_0^{\frac{1}{2q}}(1-2qt)(1-t)\,{}_0d_qt+\int_0^1(2qt-1)(1-t)\,{}_0d_qt-\int_0^{\frac{1}{2q}}(2qt-1)(1-t)\,{}_0d_qt=\frac{1}{2}\frac{1+2q^3}{(1+q)(q^2+q+1)}
\end{aligned}$$

*Remark.*If we choose $q=1$ in Theorem (1), then we obtain Remark 14 in [3] see also [[3] , page 7].

Theorem 2.*Let $f:I\to R$ be a continuous function and $0<q<1$. If $|{}_aD_\alpha f|^r$ is a convex and integrable function on $I^\circ$and $p,r>1$, $1/p+1/r=1$, then the following inequality holds:*

$$\begin{aligned}
&\left|-\frac{1}{4}\left(f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right)+\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right|\le\frac{(b-a)}{8}\left(\frac{\left((2q-1)^{p+1}+1\right)(1-q)}{2q(1-q^{p+1})}\right)^{\frac{1}{p}} \\
&\times\left[\frac{1}{1+q}\left[\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_aD_q(f)(a)\right|^r\right]^{\frac{1}{r}}+\frac{1}{1+q}\left[\left|{}_aD_q(f)(b)\right|^r+q\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r\right]^{\frac{1}{r}}\right].
\end{aligned}$$

*Proof.*Since $|{}_aD_\alpha f|^r$ is a convex function on I°, by the using Lemma 1 and using the well known Hölder inequality, and we have

$$\left|\frac{1}{4}\left(f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right)-\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\right|\leq\frac{(b-a)}{8}\left[\int_0^1|1-2qt|\left|{}_aD_q(f)\left(\frac{a+b}{2}t+(1-t)a\right)\right.\right.$$
$$\left.\left.+{}_aD_q(f)\left(bt+(1-t)\frac{a+b}{2}\right)\right|{}_0d_qt\right]\leq\frac{(b-a)}{8}\left(\int_0^1(|1-2qt|)^p\ {}_0d_qt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left(\left|{}_aD_q(f)\left(\frac{a+b}{2}t+(1-t)a\right)\right|\right)^r{}_0d_qt\right)^{\frac{1}{r}}+\frac{(b-a)}{8}\left(\int_0^1(|1-2qt|)^p\ {}_0d_qt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left(\left|{}_aD_q(f)\left(bt+(1-t)\frac{a+b}{2}\right)\right|\right)^r{}_0d_qt\right)^{\frac{1}{r}}$$
$$=\frac{(b-a)}{8}\left(\frac{\left((2q-1)^{p+1}+1\right)(1-q)}{2q(1-q^{p+1})}\right)^{\frac{1}{p}}\times\left[\frac{1}{1+q}\left[\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_aD_q(f)(a)\right|^r\right]^{\frac{1}{r}}\right.$$
$$\left.+\frac{1}{1+q}\left[\left|{}_aD_q(f)(b)\right|^r+q\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r\right]^{\frac{1}{r}}\right]$$

where

$$\int_0^1(|1-2qt|)^p\ {}_0d_qt=\int_0^{\frac{1}{2q}}(1-2qt)^p\ {}_0d_qt+\int_{\frac{1}{2q}}^1(2qt-1)^p\ {}_0d_qt$$
$$=\int_0^{\frac{1}{2q}}(1-2qt)^p\ {}_0d_qt+\int_0^1(2qt-1)^p\ {}_0d_qt-\int_0^{\frac{1}{2q}}(2qt-1)^p\ {}_0d_qt$$
$$=\frac{\left((2q-1)^{p+1}+1\right)(1-q)}{2q(1-q^{p+1})}$$

and since $\left|{}_aD_qf\right|^r$ is convex, we have

$$\int_0^1\left(\left|{}_aD_q(f)\left(\frac{a+b}{2}t+(1-t)a\right)\right|\right)^r{}_0d_qt\leq\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r\int_0^1t\ {}_0d_qt+\left|{}_aD_q(f)(a)\right|^r\int_0^1(1-t)\ {}_0d_qt$$
$$=\frac{1}{1+q}\left[\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_aD_q(f)(a)\right|^r\right]$$

with the same way,

$$\int_0^1\left(\left|{}_aD_q(f)\left(bt+(1-t)\frac{a+b}{2}\right)\right|\right)^r{}_0d_qt\leq\frac{1}{1+q}\left[\left|{}_aD_q(f)(b)\right|^r+q\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r\right]$$

Corollary 1. *If we choose $q=1$ in Theorem 2, then we obtain Corollary 16 in [3], see also [[3], page 9].*

Theorem 3. *Let $f:I\to R$ be a continuous function and $0<q<1$. If $\left|{}_aD_\alpha f\right|^r$ is a convex and integrable function on I° and $r\geq 1$, then the following inequality holds:*

$$\left|-\frac{1}{4}\left(f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right)+\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\right|$$
$$\leq\frac{(b-a)}{8}\left(\frac{q}{1+q}\right)^{1-\frac{1}{r}}\left(\frac{1}{2(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}\times\left[\left((2q^2+2q-1)\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r+(1+2q^3)\left|{}_aD_q(f)(a)\right|^r\right)^{\frac{1}{r}}\right.$$
$$\left.+\left((2q^2+2q-1)\left|{}_aD_q(f)(b)\right|^r+(1+2q^3)\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r\right)^{\frac{1}{r}}\right].$$

*Proof.*Since $|{}_aD_\alpha f|^r$ is a convex function on I°, by the using Lemma 1 and using the well known power-mean inequality, and we have

$$\left|-\frac{1}{4}\left(f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right)+\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\right| \leq \frac{(b-a)}{8}\left(\int_0^1 |1-2qt|\ {}_0d_qt\right)^{1-\frac{1}{r}}$$
$$\times\left[\left(\int_0^1 |1-2qt|\left|{}_aD_q(f)\left(\frac{a+b}{2}t+(1-t)a\right)\right|^r {}_0d_qt\right)^{\frac{1}{r}}+\left(\int_0^1 |1-2qt|\left|{}_aD_q(f)\left(bt+(1-t)\frac{a+b}{2}\right)\right|^r {}_0d_qt\right)^{\frac{1}{r}}\right]$$
$$\leq \frac{(b-a)}{8}\left(\frac{q}{1+q}\right)^{1-\frac{1}{r}}\times\left[\left(\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r\int_0^1 |1-2qt|\,t\ {}_0d_qt\ +\left|{}_aD_q(f)(a)\right|^r\int_0^1 |1-2qt|(1-t)\ {}_0d_qt\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\left|{}_aD_q(f)(b)\right|^r\int_0^1 |1-2qt|\,t\ {}_0d_qt\ +\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r\int_0^1 |1-2qt|(1-t)\ {}_0d_qt\right)^{\frac{1}{r}}\right]$$

$$\frac{(b-a)}{8}\left(\frac{q}{1+q}\right)^{1-\frac{1}{r}}\left(\frac{1}{2(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}\times\left[\left(\left(2q^2+2q-1\right)\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r+\left(1+2q^3\right)\left|{}_aD_q(f)(a)\right|^r\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\left(2q^2+2q-1\right)\left|{}_aD_q(f)(b)\right|^r+\left(1+2q^3\right)\left|{}_aD_q(f)\left(\frac{a+b}{2}\right)\right|^r\right)^{\frac{1}{r}}\right].$$

Corollary 2.*If we choose $q=1$ in Theorem 3, then we obtain Corollary 19 in [3], see also [[3], page 11].*

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