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Unimodular matrix on shallow water wave theory. Unimodularity through matrix method

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Abstract: The matrix method used in many studies of wave theory, for easy calculations, is described and used in the shallow water breaking theory. We deal with shallow water wave breaking and the method of unimodular matrix. Introducing a class of nonlinear water waves, the solution in the stratification regions using matrix method through unimodularity is given.

In this paper we present the stratification approximated by n-layers. Our method firstly involves a 2x2 matrix. Taking the production of n 2x2 matrices we choose the layers with linear variation. The main part of our work covers the fact that energy conservation law is satisfied. For this the unimodularity of the matrices is used. The models are tested against experiments concerning periodic wave transformation. The density and the speed of waves vary exponentially with depth. We conclude with experiments and some important conclusions.

Keywords: Shallow water waves, wave breaking, unimodular matrix, approximation problem.

1 Introduction

The shallow water wave is studied as a sinusoidal wave of wavelength L, height H and period T, propagating on water with undisturbed depth h. We study more specifically the propagation of nonlinear water waves. For this we calculate the approximation of the structure stratification by a set of layers in uniform way using for this matrices to connect the water region across the layers (see 2-4). We use 2 x 2 matrix to calculate the reflection and braking of water waves as a summary of the variation methods used in incident waves [7]). We formulate the problem using the real elements of matrix, saving the time of computations. To guarantee energy conservation we use the unimodularity of the matrices as we must see in this paper.

The comparison between theory and experiments represents the opportunity to reveal the experimental constraints (Farhat [8], Porter and Newman [9]). These subjects are of great interest for engineers, mathematicians and physicist, as well as the theoretical assumptions make models different from reality.

2 Main theory

2.1 Surface wave theory

We consider an irrotational flow, with velocity u, which can be expressed as: $u = \nabla\phi$, where ϕ represents a scalar potential. If we consider mass conservation, the scalar potential satisfies Laplace's equation: $\nabla\phi = 0$. It is a reasonable starting point for shallow water waves, which are greatly influenced by viscosity, surface tension or turbulence. Such a theory is based

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on some basic assumptions that establish the foundation of the following development. Let us consider a surface of the water.

$$S(x,y,t) = z - \eta(x,y,t) = 0 \tag{1}$$

Then, considering that the surface S moves with the water and always contains the same particles, we can express the zero exchange of particles by means of the material derivative:

$$\frac{DS}{Dt} = \left(\frac{\partial S}{\partial t} + u\nabla S\right) = 0 \tag{2}$$

2.2 The propagation of the shallow water waves problem

Our interesting problem is the propagation of the shallow water waves in a coastal region with some obstacles inside it. The velocity potential function φ and the surface displacement η should satisfy the Laplace equation (volume conservation):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 0 \tag{3}$$

and the free surface boundary conditions expressed in the following equations:

$$\frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x},\ \ z = \eta$$

$$\frac{\partial \varphi}{\partial t} + \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right], \tag{4}$$

3 The Mathematical models of surface waves

3.1 The linearized equation of the velocity potential function ϕ

. In this article, we mention the model developed by Massel [11], the mathematical models of surface waves that gives results for nonlinear conditions. We mention the fact that the application was study by Ohyama [12]. Among the earliest models, which solve the linearized equations, one can mention Mei and Black [13], who presents a variation of formulation of Miles [14]. Thus, we propose a model of nonlinear propagation expressed as a nonlinear wave equation. The unimodular matrix method is used to obtain responses of approximation solution on braking water waves equation.

In addition, a nonlinear source term in the right hand yields a non-homogeneous partial differential equation.

$$\left(\frac{\partial^2 \varphi}{\partial t^2} + \beta\frac{\partial \varphi}{\partial t} - c^2(x)\frac{\partial^2 \varphi}{\partial x^2}\right) = \alpha\,(\mathrm{x})\,\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial x}\right) \tag{5}$$

where β is constant and α(x) and c(x) are piecewise functions representing the nonlinearity and the phase velocity, respectively. We develop the wave function φ using the linearized equation, which is expressed as follows:

$$\left(\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{\rho}\nabla\varphi\nabla\rho\right) = 0 \tag{6}$$

Thus, we have $[\varphi_1] = 0$ and $[\frac{\partial}{\partial x}\ \varphi_1] = 0$ at x = 0. Taking the proposed solution developed by Massel [5], according to the boundary conditions at the infinity as:

$$\varphi_1 = \alpha e^{i(kx\ -\ \omega t)} - \alpha R e^{-i(kx\ +\ \omega t)}\ , x<0 \tag{7}$$

$$\varphi_1 = \alpha T e^{i(kx-\omega t)}, x>0 \tag{8}$$

In the zx plane, solution of the above equation (6) has the form;

$$\varphi(\mathrm{z,x,t}) = e^{i(Kx-\omega t)}\varphi(\mathrm{z}) \tag{9}$$

where ω is the angular frequency of the wave and K is the $\boldsymbol{x}$ component of the wave vector, which is a constant of the motion. For a rectangular stratification region between uniform media a and b, note that the constant *K, as the braking wave number* is: $K = \frac{\omega}{c}\sin\Phi$, where Φ is the angle of breaking waves (incidence at points a and b). From (5) and (6), the equation for φ(z) is

$$\rho\frac{d}{dz}\left(\frac{1}{\rho}\frac{d\varphi}{dz}\right) + q^2\varphi = 0 \tag{10}$$

where $\boldsymbol{q}$ is the normal component of the water wave vector ($q^2(\mathrm{z}) = \frac{\omega^2}{c^2} - \mathrm{K}^2$). Note that $q = \frac{\omega}{c}\cos\Phi$. The dispersion relation between ω and K will be referred as the function K = D(ω) defined in terms of the forms of the rectangular region in media $\boldsymbol{a}$ and $\boldsymbol{b}$.

The dispersion relation between ω and K will be referred as the function k = D (ω). The breaking wavenumbers obtained for the multiplies of the frequencies, will be denoted by:

$$D(n\omega) = \left\{ \begin{array}{c} k_n,\ \mathrm{x} < 0 \\ l_n\ ,\ \mathrm{x} > 0\ , \end{array} \right\}\ \mathrm{n} = 1,\ 2,\ \ldots \tag{11}$$

The second-order differential equation for φ (z) may be written as a pair of coupled first-order differential equations of $\varphi\eta$ and its derivative divided by the density

$$\left(\frac{1}{\rho}\frac{d\varphi\eta}{dz}\right) = \mathrm{D},\ \left(\rho\frac{dD}{dz}\right) = -q^2\varphi\eta \tag{12}$$

3.2 Method of solutions using unimodular matrix

Let see now the problem of stratification by a set of $\boldsymbol{n}$-homogeneous layers.

Here ρ and $\boldsymbol{q}$ take the constant values ρ_n and $\boldsymbol{q}_n$ in region $\mathrm{z}_n < \mathrm{z} < \mathrm{z}_{n+1}$ and the braking wave number is: $K = \frac{\omega}{c}\sin\Phi$, where Φ is the angle of breaking waves. So, the solutions of (13) in the $\boldsymbol{n}$-th layer are

$$\begin{aligned} \varphi(\mathrm{z}) &= \varphi_n \cos \mathrm{q}_n (z - \mathrm{z}_n) + \mathrm{q}_n \mathrm{D}_n sin\mathrm{q}_n (z - \mathrm{z}_n) \\ \mathrm{D}(\mathrm{z}) &= \mathrm{D}_n \cos \mathrm{q}_n (z - \mathrm{z}_n) + \mathrm{q}_n \varphi_n sin\mathrm{q}_n (z - \mathrm{z}_n) \end{aligned} \tag{13}$$

Using the fact that φ_n and D_n are continuous at $z = \mathrm{z}_n$, the continuity at z_{n+1} gives

$$\begin{aligned} \varphi_{n+1}(\mathrm{z}) &= \varphi_n \cos \mathrm{q}_n (\mathrm{z}_{n+1} - \mathrm{z}_n) + \mathrm{q}_n \mathrm{D}_n sin\mathrm{q}_n (\mathrm{z}_{n+1} - \mathrm{z}_n) \\ \mathrm{D}_{n+1}(\mathrm{z}) &= \mathrm{D}_n \cos \mathrm{q}_n (\mathrm{z}_{n+1} - \mathrm{z}_n) + \mathrm{q}_n \varphi_n sin\mathrm{q}_n (\mathrm{z}_{n+1} - \mathrm{z}_n) \end{aligned} \tag{14}$$

Thus, the vector formed from Φ_{n+1} and D_{n+1} is expressed by a matrix to the vector formed from Φ_n and D_n:

$$\begin{pmatrix} \varphi_{n+1} \\ \mathrm{D}_{n+1} \end{pmatrix} = \begin{pmatrix} \cos \mathrm{q}_n (\mathrm{z}_{n+1} - \mathrm{z}_n) & \mathrm{q}_n sin\mathrm{q}_n (z - \mathrm{z}_n) \\ -\mathrm{q}_n sin\mathrm{q}_n (z - \mathrm{z}_n) & \cos \mathrm{q}_n (\mathrm{z}_{n+1} - \mathrm{z}_n) \end{pmatrix} \begin{pmatrix} \varphi_n \\ \mathrm{D}_n \end{pmatrix} \tag{15}$$

This is an unimodular matrix as the its determinant is an unit. The matrix is real for real q_n.

The approximation in which an arbitrary stratification is represented by a set of homogeneous layers leads to unimodular matrices such as the one in (15).

4 Numerical methods on the matrices determining the approximations

4.1 Approximation based on constant unchanged properties of water waves

We discus for a large number of matrices. We chose the parameters that determine the conditions of the problem: The corresponding matrices M_1 and M_2 obtained from eq. 3.11 are as follows:

$$\begin{pmatrix} \cos q_n (z_{n+1} - z_n) & q_n sin q_n (z - z_n) \\ -q_n sin q_n (z - z_n) & \cos q_n (z_{n+1} - z_n) \end{pmatrix} \begin{pmatrix} \varphi_n \\ D_n \end{pmatrix} = M_n \begin{pmatrix} \varphi_n \\ D_n \end{pmatrix} \tag{16}$$

We have seen how to approximate matrices M_n that give $\varphi_{n+1}\ D_{n+1}$ in terms of $\varphi_n\ D_n$ in the n-layer. Let discus the problem in the region [a, b] boundaries z_1 and z_{n+1} of the stratification represented by n-layers. Under an obstacle emerged in that region, we have the values of φ and D at z_1 and z_{n+1} are given

$$\begin{aligned} \varphi_1 &= \alpha e^{i(kx - \omega t)} + \alpha R e^{-i(kx + \omega t)}, \alpha = q_a z_1, x < 0 \\ D_1 &= \alpha e^{i(kx - \omega t)} + \alpha R e^{-i(kx + \omega t)}, \alpha = q_a z_1, x > 0 \end{aligned} \tag{17}$$

$$\begin{aligned} \varphi_{n+1} &= \beta e^{i(kx - \omega t)} + \beta R e^{-i(kx + \omega t)}, \beta = q_a z_{n+1}, x < 0 \\ D_1 &= \beta e^{i(kx - \omega t)} + \beta R e^{-i(kx + \omega t)}, \beta = q_a z_{n+1}, x > 0 \end{aligned} \tag{18}$$

$$\begin{pmatrix} \varphi_{n+1} \\ D_{n+1} \end{pmatrix} = M_n \begin{pmatrix} \varphi_n \\ D_n \end{pmatrix} = M_n M_{n-1} \begin{pmatrix} \varphi_{n-1} \\ D_{n-1} \end{pmatrix} = \cdots = M_1 \begin{pmatrix} \varphi_1 \\ D_1 \end{pmatrix} \tag{19}$$

Here $M = M_n M_{n-1} M_{n-2} \ldots\ldots M_2 M_1$ a production matrix as a sequence product of n-layer's matrix. If m_{ij} is the element of 2x2 matrix, then

$$\begin{pmatrix} \varphi_{n+1} \\ D_{n+1} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \varphi_n \\ D_n \end{pmatrix} \tag{20}$$

Solving for the reflection and transmission amplitudes r and t we find

$$\begin{aligned} r &= e^{2i\alpha} \frac{q_a q_b m_{12} + m_{21} + i q_a m_{22} - i q_b m_{11}}{q_a q_b m_{12} - m_{21} + i q_a m_{22} = i q_b m_{11}} \\ t &= e^{i(\alpha - \beta)} \frac{2 i q_a (m_{11} m_{22} - m_{12} m_{21})}{q_a q_b m_{12} - m_{21} + i q_a m_{22} = i q_b m_{11}} \end{aligned} \tag{21}$$

Using now the unimodular matrix as M = $m_{11}m_{22} - m_{12}m_{21}$ above, we sure that a conservation law and a reciprocity law are both satisfied when the profile matrix is unimodular, that is det M= 1.

It is proved that unimodularity of M is necessary for energy conservation. If each layer matrix is unimodular, M will be unimodular, since the determinant of a product of matrices is equal to the product of their determinants. Thus unimodularity of the layer guarantees these laws, and is a desirable characteristic in any approximation scheme.

The continuity of φ and $\partial_x \varphi$ imposed in equations above is observable in figure 1, where the addition of the incident and reflected wave in the left region match in amplitude and slope with the transmitted wave.

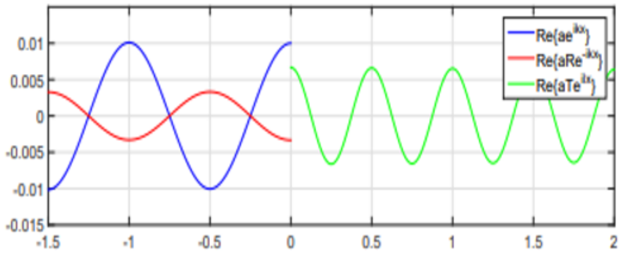

Fig. 1: In $x > 0$ a transmitted wave.(Left). Real part of wave ϕ, solution of the linear problem. (Right)

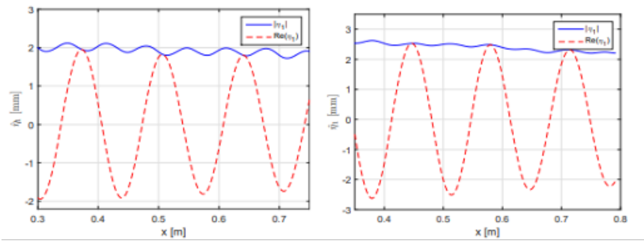

Fig. 2: The depth is $h = 2$ cm, start at $x = 0.6$ m. Reflected wave (Left), transmitted wave. (Right)

4.2 Comparison of methods

Table of comparison of 5 matrix algorithms, all using 7 matrices. The letters C, L, and U stand for constant, linear, and unirnodular.

Method	Constant	Linear	unimodular	C_∞
Average error %	2.0	0.8	0.2	1.3
Average of det M - 1	2	10^{-3}	0	0

Table 1: Comparison of five matrix algorithms, all using seven matrices for constant, linear, and unirnodular

As another problem that we will discuss in the nearest article, we will discus for a large number of matrices. We leave to construct a profile matrix for the production of n-layers matrices, solving the reflection and trasmission amplitudes of shallow water waves.

5 Conclusion

In conclusion, we analyzed the importance of the unimodular matrices. At first, we note that the method is chosen to use the unimodular matrix. Of this method we have comparable accuracy in the reflectance and are comparable in programming simplicity.

However, other methods are considerably longer to execute since the sine and cosine matrix elements take an order of magnitude longer to compute.

At first we note that the C and L methods give almost the same results when in all the C methods, the constant parameters for each layer were chosen to be the exact values at the middle of the layer. Thus the C, and L matrix elements are almost the same.

Of the three methods that have det M=1 (to within the numerical precision available), U and C_∞ have comparable accuracy in the reflectance, and are comparable in programming simplicity. However, C, is considerably longer to execute, since the sine and cosine matrix elements take an order of magnitude longer to compute.

The unimodular method based on linear variation) is our preference, because very simple matrix elements are required, for example in order r to use a small programmable calculator. We note also that the matrix method can also be used in the more usual numeric solution of the differential equation ([16]).

The great advantage of our technique is the possibility to measure a two-dimensional field η (x, y, z) with good temporal resolution. At low frequencies and sufficiently small amplitude, the reflection can be reduced many times by means of this mechanism.

It is proved that unimodularity of M is necessary for energy conservation. If each layer matrix is unimodular, M will be unimodular since the determinant of a product of matrices is equal to the product of their determinants. Thus, unimodularity of the layer guarantees these laws, and is a desirable characteristic in any approximation scheme.

In conclusion, we have formulated the problem in terms of a product of 2x2 layer matrices, shown that these matrices should be unimodular for energy conservation and methods that use unimodularity are simple to program.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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