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The numerical solutions of fractional differential equations with beta-derivative

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Abstract: In this study, the Fibonacci collocation method is investigated to obtain the numerical solution of the fractional order differential equations based on the beta-derivative. The problem is firstly reduced into an algebraic system; later the unknown coefficients of the approximate solution function are obtained. To illustrate the efficiency of the proposed method, an example is solved, and the obtained results are compared with the exact solutions.

Keywords: Collocation method, Fibonacci polynomials, Beta derivative.

1 Introduction

There are several different fractional integral and derivative definitions in the literature. In [2] authors proposed a fractional derivative definition the so-called beta-derivative. Many articles concern analytical or numerical solutions of the fractional differential equations involving beta-derivative. Therefore, in this paper, Fibonacci collocation method is proposed to obtain the numerical solutions of the following fractional order initial value problem with variable coefficients

$$Q_\beta (x) u^{(\beta)} (x) + \sum_{k=0}^{m} p_k (x) u^{(k)} (x) = g (x), \quad 0 \leq x \leq 1, \quad 0 < \beta < 1 \tag{1}$$

with the initial conditions

$$\sum_{k=0}^{m-1} \left[a_k u^{(k)} (0) + b_k u^{(k)} (0) \right] = \delta_k, \tag{2}$$

where $u^{(0)} (x) = u(x)$ and $u(x)$ is an unknown functions. $P_k (x)$, $Q_\alpha (x)$ and g(x) are given continuous functions on the interval $[0,1]$, a_k, b_k and δ_k are suitable constants. The aim of this study is to get the approximate solution as the truncated Fibonacci series defined by

$$u (x) = \sum_{n=1}^{N+1} c_n F_n (x) \tag{3}$$

where $F_n(x)$ denotes the Fibonacci polynomials; c_n $(1 \leq n \leq N+1)$ are unknown Fibonacci polynomial coefficients, and N is chosen as any positive integer such that $N \geq m$.

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2 Preliminaries

The recurrence relation of the Fibonacci polynomials is defined by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \tag{4}$$

where $F_1(x) = 1,\ F_2(x) = x$ and $n \geq 3$.

Definition 1. Let f be a function, such that $f : [a, \infty) \to \mathbb{R}$. Then the beta-derivative is defined as

$$D_x^{\beta}(f(x)) = \lim_{\mathrm{E}\to 0} \frac{f\left(x + \mathrm{E}\left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - f(x)}{\mathrm{E}}$$

for all $x \geq a,\ \beta \in (0, 1]$. If the above limit exists, then f is said to be beta-differentiable.

Theorem 1. Assume that $f : [a, \infty) \to \mathbb{R}$ is a differentiable and also beta-differentiable function. Let g be a differentiable function defined in the range of f; then the following rule satisfies

$$D_x^{\beta}((g \circ f)(x)) = \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} f'(x)\, g'(f(x)).$$

3 Fundamental Relations

We can write Fibonacci polynomials in the matrix form

$$F(x) = T(x)M \tag{5}$$

where $F(x) = [F_1(x)\ F_2(x) \ldots F_{N+1}(x)]$, $T(x) = \left(1\ x\ x^2\ x^3 \ldots . x^N\right)$, $C = \left(c_1\ c_2 \ldots\ c_{(N+1)}\right)^T$ and

$$M = \begin{bmatrix} 1&0&1&0&1&0&1&0&1&0&\ldots \\ 0&1&0&2&0&3&0&4&0&5&\ldots \\ 0&0&1&0&3&0&6&0&10&0&\ldots \\ 0&0&0&1&0&4&0&10&0&20&\ldots \\ 0&0&0&0&1&0&5&0&15&0&\ldots \\ 0&0&0&0&0&1&0&6&0&21&\ldots \\ 0&0&0&0&0&0&1&0&7&0&\ldots \\ 0&0&0&0&0&0&0&1&0&8&\ldots \\ 0&0&0&0&0&0&0&0&1&0&\ldots \\ 0&0&0&0&0&0&0&0&0&1&\ldots \\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots \end{bmatrix}$$

Then we set the approximate solutions defined by a truncated Fibonacci series (3) in the matrix for

$$u(x) = F(x)C. \tag{6}$$

By using the relations Eq.(5) and Eq.(6), the matrix relation is expressed as

$$\begin{aligned} u(x) &\cong u_N(x) = F(x)C = T(x)MC \\ u'(x) &\cong u_N'(x) = T(x)BMC \\ u''(x) &\cong u_N''(x) = T(x)B^2MC \\ &\cdots \\ u^{(k)}(x) &\cong u_N^{(k)}(x) = T(x)B^kMC \end{aligned} \tag{7}$$

Also, the relations between the matrix $T(x)$ and its derivatives $T'(x), T''(x), \ldots, T^{(k)}(x)$ are

$$\begin{gathered} T'(x) = T(x)B,\ T''(x) = T(x)B^2 \\ T'''(x) = T(x)B^3, \ldots, T^k(x) = T(x)B^k \end{gathered} \tag{8}$$

By substituting the Fibonacci collocation points defined by

$$x_i = \frac{i}{N},\ i = 0, 1, \ldots N \tag{9}$$

into Eq.(7), we have

$$u^{(k)}(x_i) = T(x_i)B^kMC. \tag{10}$$

and the compact form of the relation (10) becomes

$$U^{(k)} = TB^kMC, \qquad k = 0, 1, 2, \tag{11}$$

where

$$U^{(k)} = \begin{bmatrix} u^{(k)}(x_0) \\ u^{(k)}(x_1) \\ \vdots \\ u^{(k)}(x_N) \end{bmatrix},$$

$$\mathrm{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, B^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} T(x_0) \\ T(x_1) \\ \vdots \\ T(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \cdots & x_0^N \\ 1 & x_1 & \cdots & x_1^N \\ 1 & \vdots & \cdots & \cdots \\ 1 & x_N & \cdots & x_N^N \end{bmatrix}.$$

In addition, we can obtain the matrix forms $U^{(\beta)}$ which appears in the fractional part of Eq. (1), by using Eq.(7) as

$$U^{(\beta)} = R_\beta TBMC$$

x	N=2	N=5	N=8
0.2	1.33067×10^{-3}	2.26982×10^{-7}	8.89039×10^{-11}
0.4	1.05817×10^{-2}	4.85018×10^{-7}	1.83602×10^{-10}
0.6	3.53575×10^{-2}	8.12754×10^{-7}	2.64480×10^{-10}
0.8	8.26439×10^{-2}	8.98300×10^{-7}	3.02574×10^{-10}
1.0	1.58529×10^{-1}	3.46906×10^{-5}	1.11643×10^{-8}

Table 1: The numerical results of the absolute error function for different values of N

Substituting the collocation points ($x_i = i/N, i = 0,1,\ldots,N$) into Eq. (1), gives the equation

$$Q_\beta(x_i)u^{(\beta)}(x_i)+\sum_{k=0}^{m}p_k(x_i)u^{(k)}(x_i)=g(x_i), \qquad 0<\beta<1$$

which can be expressed as

$$\left\{Q_\beta U^{(\beta)}+\sum_{k=0}^{m}P_kU^{(k)}\right\}C=\ G$$

where

$$R_\beta=diag\left[\left(x_0+\frac{1}{\Gamma(\beta)}\right)^{1-\beta},\left(x_1+\frac{1}{\Gamma(\beta)}\right)^{1-\beta},\ldots,\left(x_N+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right].$$

$$P_k=diag\left[p_k(x_0)\ \ p_k(x_1)\ \ldots\ \ p_k(x_N)\right],$$

$$Q_\beta=diag\left[Q_\beta(x_0)\ \ Q_\beta(x_1)\ \ldots\ \ Q_\beta(x_N)\right]$$

and

$$G=\ \ [g(x_0)\ \ g(x_1)\ \ \ldots\ \ \ g(x_N)]T.$$

In this case, the fundamental matrix equation can be obtained as

$$\left\{Q_\beta R_\beta TBM+\sum_{k=0}^{m}P_kTB_kM\right\}C=G.$$

If the initial conditions applied to the equation, we obtain a linear matrix system. Thus, by solving the linear matrix system, the unknown Fibonacci coefficients are calculated and the Fibonacci polynomial solutions is found.

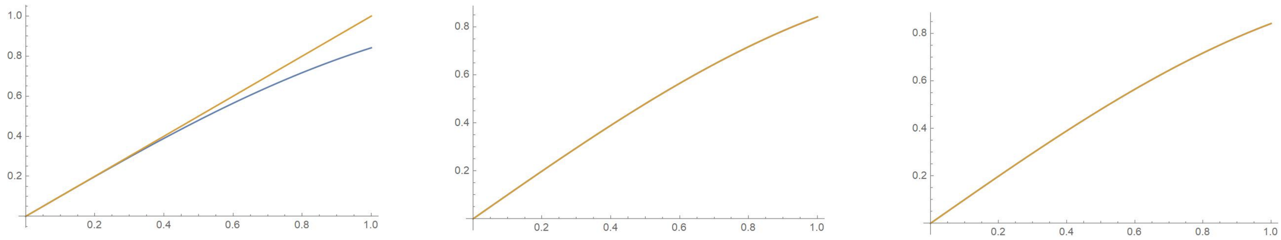
4 Application

In this section, a numerical example is presented to illustrate the efficient of the proposed method. On these problems, the method is tested by using the absolute error function. The obtained numerical results are presented with tables and graphics.

Example 1. Assume that the following differential equation

$$u^{''}(x)+xu^{(0.5)}(x)+u(x)=x\left(\frac{1}{\Gamma(0.5)}+x\right)^{0.5}\cos(x)\ ;\quad u(0)=0,u^{'}(0)=1$$

The exact solution of the equation is given by $u(x)=\sin(x)$. Table 1 presents the numerical results of the absolute error function for different values of N. Also, in Figure 1, it is presented that the graphics of the approximate and exact solutions for different values of N.



(a) $N = 2$. **(b)** $N = 5$. **(c)** $N = 8$.

Fig. 1: The graphics of the approximate and exact solutions for different values of N.

5 Conclusion

In this study, Fibonacci collocation method is applied to obtain the approximate solutions of fractional differential equations constructed with beta derivative. Illustrative example is given to show and support our findings. Regarding to the findings it can be said that Fibonacci collocation method is an efficient method to obtain the approximate solutions of the given class of fractional differential equations.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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