Almost $C(\alpha)$-manifold on $M$–projective curvature tensor

Tuğba Mert$^1$ and Mehmet Atçeken$^2$

$^1$Department of Mathematics, University of Sivas Cumhuriyet, Sivas, Turkey
$^2$Department of Mathematics, University of Aksaray, Aksaray, Turkey

Received: 16 February 2022, Accepted: 8 June 2022
Published online: 17 August 2022.

Abstract: In this article, the behavior of the $C(\alpha)$-manifold satisfying the conditions $R(X,Y)W^* = 0, W^*(X,Y)R = 0, W^*(X,Y)\tilde{Z} = 0, W^*(X,Y)S = 0$ and $W^*(X,Y)\tilde{C} = 0$ on the $M$–projective curvature tensor is investigated. The $C(\alpha)$–Manifold is characterized according to these states of the curvature tensor. Here, $W^*, R, S, \tilde{Z}$ and $\tilde{C}$ are $M$–projective, Riemann, Ricci, concircular and quasi-conformal curvature tensors.

Keywords: $M$–Projective Curvature Tensor, Ricci Curvature Tensor, Concircular Curvature Tensor

1 Introduction

A new tensor field

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4n} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$

(1)

is defined by Pokhariyal and Mishra in $n$–dimensional Riemannian manifolds [1]. The $W^*$ tensor field is called the $M$–projective tensor field where $Q$ is the Ricci operator and $S$ is the Ricci tensor. The definition and properties of the $M$–projective curvature tensor are given by Ojha in Sasakian and Kaehler manifolds [2],[3]. In recent years, many geometers have worked on the $M$–projective curvature tensor [4]-[10]. Again, many authors have worked on curvature tensors in almost $C(\alpha)$–manifold [11]-[13].

Based on the many studies mentioned above, in this article, the curvature conditions of $C(\alpha)$–manifold $R(X,Y)W^* = 0, W^*(X,Y)R = 0, W^*(X,Y)\tilde{Z} = 0, W^*(X,Y)S = 0$ and $W^*(X,Y)\tilde{C} = 0$ are searched.

Let’s take an $(2n+1)$–dimensional differentiable $M$ manifold. If it admits a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying the following conditions;

$$\phi^2X = -X + \eta(X)\xi \text{ and } \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \text{ and } g(X, \xi) = \eta(X),$$

for all $X, Y \in \chi(M)$ and $\xi \in \chi(M), (\phi, \xi, \eta, g)$ is called almost contact metric structure and $(M, \phi, \xi, \eta, g)$ is called almost contact metric manifold. On the $(2n+1)$ dimensional $M$ manifold,

$$g(\phi X, Y) = -g(X, \phi Y),$$

* Corresponding author e-mail: tmert@cumhuriyet.edu.tr

© 2022 BISKA Bilisim Technology
for all \(X, Y \in \chi(M)\), that is, \(\phi\) is an anti-symmetric tensor field according to the \(g\) metric. The transformation \(\Phi\) defined as
\[
\Phi(X, Y) = g(X, \phi Y),
\]
for all \(X, Y \in \chi(M)\), is called the fundamental 2-form of the \((\phi, \xi, \eta, g)\) almost contact metric structure, where
\[
\eta \wedge \Phi^c \neq 0.
\]

If the \(R\) Riemann curvature tensor of the \(M\) almost contact metric manifold satisfies the condition
\[
R(X, Y, Z, W) = R(X, Y, \phi Z, \phi W) + \alpha \{ -g (X, Z) g (Y, W) + g (X, W) g (Y, Z) + g (X, \phi Z) g (Y, \phi W) - g (X, \phi W) g (Y, \phi Z) \},
\]
for all \(X, Y, Z, W \in \chi(M), \exists \alpha \in \mathbb{R}\), then \(M\) is called the almost \(C(\alpha)\)-manifold. Also, the Riemann curvature tensor of a almost \(C(\alpha)\)-manifold with \(c\)-constant sectional curvature is given by
\[
R(X, Y) Z = \left(\frac{c + 3\alpha}{4}\right) \{ g (Y, Z) X - g (X, Y) Z \} + \left(\frac{c - \alpha}{4}\right) \{ g (X, \phi Z) \phi Y - g (Y, \phi Z) \phi X \}
+ 2g(X, \phi Y) \phi Z + \eta(\eta) \eta(X) X + g(X, Z) \eta(\eta) \xi - g(Y, Z) \eta(X) \xi \}.
\]

For a \((2n + 1)\)-dimensional \(M\) almost \(C(\alpha)\)-manifold, the following equations are provided.
\[
S(X, Y) = \left[\frac{\alpha (3n - 1) + c(n + 1)}{2}\right] g(X, Y) + \frac{\alpha - c}{2} \eta(X) \eta(Y),
\]
\[
S(X, \xi) = 2n \alpha \eta(X),
\]
\[
QX = \left[\frac{\alpha (3n - 1) + c(n + 1)}{2}\right] X + \frac{\alpha - c}{2} \eta(X) \xi,
\]
\[
Q\xi = 2n \alpha \xi,
\]
\[
Q\phi Y = \frac{r - 2n \alpha}{2n} QY,
\]
for all \(X, Y \in \chi(M)\), where \(Q\) and \(S\) are the Ricci operator and Ricci tensor of manifold \(M\), respectively.

\section{2 \(C(\alpha)\)-manifolds satisfying some important conditions on the \(M\)-projective curvature tensor}

Let \(M\) be a \((2n + 1)\)-dimensional almost \(C(\alpha)\)-manifold and \(R\) be the Riemann curvature tensor of \(M\) manifold. So, if we choose \(X = \xi\) in (2), we get
\[
R(\xi, Y) Z = \alpha [g(Y, Z) \xi - \eta(Z) Y].
\]
Similarly, if we choose \(Z = \xi\) in (2), we get
\[
R(X, Y) \xi = \alpha [\eta(Y) X - \eta(X) Y].
\]
In addition, if \( Y = \xi \) is chosen in (9),
\[
R(X, \xi) \xi = \alpha [X - \eta (X) \xi]
\]
is obtained. If the inner product of both sides of (2) is taken by \( \xi \in \chi (M) \), we have
\[
\eta (R(X, Y)Z) = \alpha [g(Y, Z) \eta (X) - g(X, Z) \eta (Y)].
\]
Finally, if we choose \( X = \xi \) in the (1), then it reduces the form
\[
W^* (\xi, Y) Z = \frac{(n+1)(\alpha-c)}{8n} [g(Y, Z) \xi - \eta (Z) Y],
\]  
and if we choose \( Z = \xi \) in the same equation, we get
\[
W^* (X, Y) \xi = \frac{(n+1)(\alpha-c)}{8n} [\eta (Y)X - \eta (X) Y].
\]

**Theorem 1.** Let \( M \) be a \((2n+1)\)-dimensional almost \( C (\alpha) \)–manifold. If \( M \) is \( M \)–projective flat, then \( M \) is an Einstein manifold.

**Proof.** Let’s assume that manifold \( M \) is \( M \)-projective flat. From (1), we can write
\[
W^* (X, Y) Z = 0,
\]
for each \( X, Y, Z \in \chi (M) \). Then from (1), we obtain
\[
R(X, Y) Z = \frac{1}{4n} [S(Y, Z)X - S(X, Z) Y + g(Y, Z) QX - g(X, Z) QY],
\]
for each \( X, Y, Z \in \chi (M) \). If we choose \( Z = \xi \) in (11) and using (4), (9), we obtain
\[
\frac{\alpha}{2} [\eta (Y)X - \eta (X) Y] = \frac{1}{4n}[\eta (Y)QX - \eta (X) QY].
\]
In the last equation, if we first choose \( X = \xi \) and we take inner product both sides of the last equation by \( Z \in \chi (M) \), then we get
\[
S(Y, Z) = 2n\alpha g(Y, Z)
\]
It is clear from the last equation that \( M \) is Einstein manifold.

**Theorem 2.** Let \( M \) be \((2n+1)\)–dimensional a \( C (\alpha) \)–manifolds. Then \( W^* (X, Y) R = 0 \) if and only if either the scalar curvature of \( M \) is \( r = 2n\alpha (2n+1) \) or \( M \) reduces real space form with constant sectional curvature.

**Proof.** Suppose that \( W^* (X, Y) R = 0 \). Then, we have
\[
\]
If we choose \( X = \xi \) in here, we get
\[
(W^* (\xi, Y)) R(U, V, Z) = W^* (\xi, Y) R(U, V) Z - R(W^* (\xi, Y) U, V) Z - R(U, W^* (\xi, Y) V) Z - R(U, V) W^* (\xi, Y) Z = 0, \tag{12}
\]
for each \( Y, U, V, Z \in \mathcal{X}(M) \). In (12), using (10), we obtain
\[
\frac{(n + 1)(\alpha - c)}{8n} [g(Y, R(U, V) Z) \xi - \eta(R(U, V) Z) Y - g(Y, U) R(\xi, V) Z + \eta(U) R(\xi, V) Z - g(Y, V) R(U, \xi) Z + \eta(V) R(U, \xi) Z - g(Y, Z) R(U, V) \xi + \eta(Z) R(U, V) Y] = 0. \tag{13}
\]
Substituting \( U = \xi \) in (13) and using (8), (9), we conclude
\[
\frac{(n + 1)(\alpha - c)}{8n} [R(Y, V) Z - \alpha (g(V, Z) Y - g(Y, Z) V)] = 0. \tag{14}
\]
From (14), we have
\[
c = \alpha. \tag{15}
\]
In addition, since the scalar curvature of a \( C(\alpha) \)-manifold with constant sectional curvature is
\[
r = n[\alpha (3n + 1) + c(n + 1)] \tag{16}
\]
if the expression (15) is also put in (16), we get
\[
r = 2n\alpha (2n + 1). \]
On the other hand, from (14) we get
\[
R(Y, V) Z = \alpha [g(V, Z) Y - g(Y, Z) V].
\]
Thus, \( M \) is reduced to the real space form with constant sectional curvature. The converse is obvious and the proof is completed.

Let \( M \) be a \((2n + 1)\)-dimensional Riemannian manifold. Then the conircular curvature tensor \( \tilde{Z} \) is defined as
\[
\tilde{Z}(X, Y) Z = R(X, Y) Z - \frac{r}{2n(2n + 1)} [g(Y, Z) X - g(X, Z) Y], \tag{17}
\]
for all \( X, Y, Z \in \mathcal{X}(M) \). If we choose \( X = \xi \) in (17), we get
\[
\tilde{Z}(\xi, Y) Z = \left( \alpha - \frac{r}{2n(2n + 1)} \right) [g(Y, Z) \xi - \eta(Z) Y], \tag{18}
\]
and when we choose \( Z = \xi \) in (18) we get
\[
\tilde{Z}(\xi, Y) \xi = \left( \alpha - \frac{r}{2n(2n + 1)} \right) [\eta(Y) \xi - Y].
\]

**Theorem 3.** Let \( M \) be \((2n + 1)\)-dimensional \( C(\alpha) \)-manifold. Then \( W^* (X, Y) \tilde{Z} = 0 \) if and only if either the scalar curvature of \( M \) is \( r = 2n\alpha (2n + 1) \) or \( M \) reduces real space form with constant sectional curvature-c.
Proof. Suppose that $W^* (X,Y) \tilde{Z} = 0$. Then we have

$$(W^* (X,Y) \tilde{Z}) (U,V,Z) = W^* (X,Y) \tilde{Z} (U,V) Z - \tilde{Z} (W^* (X,Y) U,V) Z - \tilde{Z} (U,W^* (X,Y) V) Z - \tilde{Z} (U,V) W^* (X,Y) Z = 0.$$  

If we choose $X = \xi$ in here, we get

$$(W^* (\xi,Y) \tilde{Z}) (U,V,Z) = W^* (\xi,Y) \tilde{Z} (U,V) Z - \tilde{Z} (W^* (\xi,Y) U,V) Z - \tilde{Z} (U,W^* (\xi,Y) V) Z - \tilde{Z} (U,V) W^* (\xi,Y) Z = 0, \tag{19}$$

for each $Y,U,V,Z \in \chi (M)$. In (19), using (10), we obtain

$$\frac{(n+1)(\alpha-c)}{8n} \left[ g (Y, \tilde{Z} (U,V) Z) \xi - \eta (\tilde{Z} (U,V) Z) Y \right. \\
- g (Y,U) \tilde{Z} (\xi,V) Z + \eta (U) \tilde{Z} (Y,V) Z - g (Y,V) \tilde{Z} (U,\xi) Z \\
\left. + \eta (V) \tilde{Z} (U,Y) Z - g (Y,Z) \tilde{Z} (U,V) \xi + \eta (Z) \tilde{Z} (U,V) Y \right] = 0. \tag{20}$$

Taking $U = \xi$ in (20) and using (18), we obtain

$$\frac{(n+1)(\alpha-c)}{8n} \left[ \tilde{Z} (Y,V) Z - \left( \alpha - \frac{r}{8n(2n+1)} \right) \right. \\
\left. (g (V,Z) Y - g (Y,Z) V) \right] = 0. \tag{21}$$

In (21), using (17) we conclude

$$\frac{(n+1)(\alpha-c)}{8n} [\tilde{R} (Y,Z) V - \alpha (g (V,Z) Y - g (Y,Z) V) ] = 0.$$  

This proves our assertion. The converse obvious.

Theorem 4. Let $M$ be $(2n+1)$–dimensional a $C (\alpha)$–manifold. Then $W^* (X,Y) S = 0$ if and only if either the scalar curvature of $M$ is $r = 2n\alpha (2n+1)$ or $M$ reduces an Einstein manifold.

Proof. Suppose that $W^* (X,Y) S = 0$. Then we can easily see that

$$S (W^* (X,Y) Z, U) + S (Z, W^* (X,Y) U) = 0.$$  

If we choose $X = \xi$ in here, we get

$$S (W^* (\xi,Y) Z, U) + S (Z, W^* (\xi,Y) U) = 0. \tag{22}$$

In (22), using (10), we obtain

$$\frac{(n+1)(\alpha-c)}{8n} [2n\alpha \eta (U) g (Y,Z) - \eta (Z) S (Y,U) + 2n\alpha \eta (Z) g (Y,U) - \eta (U) S (Z,Y) ] = 0. \tag{23}$$

Substituting $Z = \xi$ in (23), we find

$$\frac{(n+1)(\alpha-c)}{8n} [-S (Y,U) + 2n\alpha g (Y,U) ] = 0. \tag{24}$$
From (24), we get
\[ c = \alpha. \]
This tells us that the scalar curvature of \( M \) is
\[ r = 2n\alpha (2n + 1). \]
On the other hand, from (24) we have
\[ S(Y, U) = 2n\alpha g(Y, U), \]
which implies \( M \) reduces an Einstein manifold. This proves our assertion. The converse is obvious.

The concept of the quasi-conformal curvature tensor was defined by Yano and Sowaki as
\[
\hat{C}(X, Y) Z = aR(X, Y) Z + b[S(Y, Z) X - S(X, Z) Y + g(Y, Z) QX - g(X, Z) QY]
- \frac{r}{2n + 1} \left[ \frac{1}{2} \left( \frac{a}{2n} + 2b \right) \right] [g(Y, Z) X - g(X, Z) Y],
\]
where \( a \) and \( b \) are constants, \( Q \) is the Ricci operator, \( S \) is the Ricci tensor and \( r \) is the scalar curvature of the manifold. If \( \hat{C} = 0 \), then this manifold is called a quasi-conformal flat. If \( X = \xi \) is chosen in (25),
\[
\hat{C}(\xi, Y) Z = \left[ \frac{bc(n + 1) + a(2a + 7b - b)}{2} \right] - \frac{r}{2n + 1} \left[ \frac{1}{2} \left( \frac{a}{2n} + 2b \right) \right] [g(Y, Z) \xi - \eta(Z) Y],
\]
and if \( Z = \xi \) is chosen in (26), we reach at
\[
\hat{C}(\xi, \xi) = \left[ \frac{a(2a + 7b - b)}{2} \right] - \frac{r}{2n + 1} \left[ \frac{1}{2} \left( \frac{a}{2n} + 2b \right) \right] [\eta(Y) \xi - \eta(Y) \xi] + b[2n\alpha \eta(Y) \xi - QY].
\]

**Theorem 5.** Let \( M \) be \((2n + 1)\)-dimensional a \( C(\alpha) \)-manifolds. Then \( W^*(X, Y) \hat{C} = 0 \) if and only if either the scalar curvature of \( M \) is \( r = 2n\alpha (2n + 1) \) or \( M \) reduces real space form with constant sectional curvature.

**Proof.** Suppose that \( W^*(X, Y) \hat{C} = 0 \). Then, we have
\[
(W^*(X, Y) \hat{C})(U, V, Z) = W^*(X, Y) \hat{C}(U, V) Z - \hat{C}(W^*(X, Y) U, V) Z
- \hat{C}(U, W^*(X, Y) V) Z - \hat{C}(U, V) W^*(X, Y) Z = 0.
\]
If we choose \( X = \xi \) in here
\[
(W^*(\xi, Y) \hat{C})(U, V, Z) = W^*(\xi, Y) \hat{C}(U, V) Z - \hat{C}(W^*(\xi, Y) U, V) Z
- \hat{C}(U, W^*(\xi, Y) V) Z - \hat{C}(U, V) W^*(\xi, Y) Z = 0,
\]

© 2022 BISKA Bilisim Technology
for each $Y, U, V, Z \in \chi(M)$. Using (10) in (28), we get
\[
\frac{(n + 1)(\alpha - c)}{8n} g(Y, \tilde{C}(U, V, Z) \xi - \eta(\tilde{C}(U, V) Z Y
\]
\[-g(Y, U) \tilde{C}(\xi, V) Z + \eta(U) \tilde{C}(Y, V) Z - g(Y, V) \tilde{C}(U, \xi) Z
\]
\[+ \eta(V) \tilde{C}(U, Y) Z - g(Y, Z) \tilde{C}(U, V) \xi + \eta(Z) \tilde{C}(U, V) Y \right] = 0.
\] (29)

Taking $U = \xi$ in (29) and using (26), we obtain
\[
\left[ \frac{(n + 1)(\alpha - c)}{8n} \right] \otimes \left\{ \tilde{C}(Y, Z) V - \left[ \frac{bc(n + 1) + \alpha(2a + 7bn - b)}{2} - \frac{r}{2n + 1} \left( \frac{a}{2n + 2b} \right) \right] \right. 
\]
\[= g(V, Z) Y - g(V, Z) V \right\} = 0.
\]

In the last equation, if (25) is written in its place and necessary adjustments are made, we get
\[
aR(Y, V) Z = \left[ \frac{\alpha(2a + bn + b) - bc(n + 1)}{2} \right] g(V, Z) Y - g(V, Z) V \]
\[- \frac{b(\alpha - c)(n + 1)}{2} \left[ \eta(V) \eta(X) Y - \eta(Y) \eta(Z) V + g(V, Z) \eta(Y) \xi - g(Y, Z) \eta(V) \xi \right].
\] (30)

Substituting $Y \rightarrow \phi Y$ and $V \rightarrow \phi V$ in (30), we conclude
\[
R(\phi Y, \phi V) Z = \left[ \frac{\alpha(2a + bn + b) - bc(n + 1)}{2} \right] g(V, Z) Y - g(V, Z) V.
\]

This proves our assertion. The converse is obvious.

References


