New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2022.494

On the Shortest Distance in the Plane $\mathbb{R}^2_{\pi 3}$

Z. Akça & S. Nazlı

Eskişehir Osmangazi University, Department of Mathematics and Computer Science, 26480 Eskişehir, Turkey.

Received: 18 October 2022 , Accepted: 2 December 2022
Published online: 20 December 2022.

Abstract: In this paper, we study the shortest distance of a point to the line and area of a triangle in Iso-taxicab Geometry.

Keywords: Distance; non-Euclidean geometry.

1 Introduction

The Minkowski distance is a metric in a normed vector space which can be considered as a generalization of both the Euclidean distance and the taxicab distance. Taxicab geoemetry is a geometry whose usual distance function or metric of Euclidean geometry is replaced by a new metric in which the distance between two points is the sum of the absolute differences of their cartesian coordinates in [4], [8]. Iso-taxicab geometry is a non-Euclidean geometry defined by K. O. Sowell.in 1989 in [7]. In this geometry presented by Sowell three distance functions arise depending upon the relative positions of the points A and B. There are three axes at the origin; the x-axis, the y-axis and the y' -axis, having 60^0 angle which each other. These tree axes separate the plane into six regions. The iso-taxicab trigonometric functions in iso-taxicab plane with three axes were given in [5], [6]. A family of distances, $d_{\pi n}$, that includes Taxicab, Chinese-Checker and Iso-taxi distances, as special cases introduced and the group of isometries of the plane with $d_{\pi n}$ metric is the semi-direct product of D_{2n} and $T(2)$ was shown in [3]. The trigonometric functions in $\mathbb{R}^2_{\pi 3}$ and the versions in the plane $\mathbb{R}^2_{\pi 3}$ of some Euclidean theorems were given in [1], [2].

The definition of $d_{\pi n}-$distances family is given as follows;

Definition 1. *Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be any two points in $\mathbb{R}^2$, a family of $d_{\pi n}$ distances is defined by*

$$d_{\pi n}(A,B) = \frac{1}{\sin\frac{\pi}{n}}\left(\left|\sin\frac{k\pi}{n} - \sin\frac{(k-1)\pi}{n}\right||x_1-x_2| + \left|\cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n}\right||y_1-y_2|\right)$$
$$\text{where}\begin{cases} 1\le k\le\left[\frac{n-1}{2}\right], k\in\mathbb{Z} & , \tan\frac{(k-1)\pi}{n}\le\left|\frac{y_2-y_1}{x_2-x_1}\right|\le\tan\frac{k\pi}{n} \\ k=\left[\frac{n+1}{2}\right] & , \tan\frac{\left[\frac{n-1}{2}\right]\pi}{n}\le\left|\frac{y_2-y_1}{x_2-x_1}\right|<\infty \text{ or } x_1=x_2 \end{cases}$$

For $n = 3$ and $k = 1$ or $k = 2$, we obtain the formula of $d_{\pi 3}$-distance between the points A and B according to the inclination in the plane $\mathbb{R}^2_{\pi 3}$ as the following:

$$d_{\pi 3}(A,B) = \frac{1}{\sin\frac{\pi}{3}}\left(\left|\sin\frac{k\pi}{3} - \sin\frac{(k-1)\pi}{3}\right||x_1-x_2| + \left|\cos\frac{(k-1)\pi}{3} - \cos\frac{k\pi}{3}\right||y_1-y_2|\right)$$
$$\text{where}\begin{cases} k=1 & , 0\le\left|\frac{y_2-y_1}{x_2-x_1}\right|\le\tan\frac{\pi}{3} \\ k=2 & , \tan\frac{\pi}{3}\le\left|\frac{y_2-y_1}{x_2-x_1}\right|<\infty \text{ or } x_1=x_2 \end{cases}$$

© 2022 BISKA Bilisim Technology
* Corresponding author e-mail: e-mail: zakca@ogu.edu.tr

or

$$d_{\pi 3}(A,B)=\begin{cases}|x_1-x_2|+\frac{1}{\sqrt{3}}|y_1-y_2| &, 0\leq\left|\frac{y_2-y_1}{x_2-x_1}\right|\leq\sqrt{3}\\ \frac{2}{\sqrt{3}}|y_1-y_2| &, \sqrt{3}\leq\left|\frac{y_2-y_1}{x_2-x_1}\right|<\infty \text{ or } x_1=x_2.\end{cases}$$

2 Distance and Area in the plane $\mathbb{R}^2_{\pi 3}$

Now we give the shortest distance of a point to the line and area of a triangle to the plane $\mathbb{R}^2_{\pi 3}$.

Theorem 1. *The shortest distance of a point $P_0=(x_0,y_0)$ to the line l given by*

$$ax+by+c=0$$

in the plane $\mathbb{R}^2_{\pi 3}$ is

$$d_{\pi 3}(P_0,l)=\rho(\frac{-1}{m})d_E(P_0,l).$$

Proof. Let X be the point where the line segment drawn from the point P_0 to the line l touches the line l. The line segment $[PX]$ length is the shortest distance from the point P_0 to line l. Also, if the slope of the line l is m then the slope of the line segment $[PX]$ is $(\frac{-1}{m})$. The shortest distance from the point P_0 to the line l in the plane $\mathbb{R}^2_{\pi 3}$ is

$$d_{\pi 3}(P_0,l)=\rho(\frac{-1}{m})d_E(P_0,l).$$

If calculations are made for this equation, the shortest distance from the point P_0 to the line l in the plane $\mathbb{R}^2_{\pi 3}$ is

$$d_{\pi_3}(P_0,l)=\begin{cases}\left(\frac{1}{\sqrt{1+(\frac{-1}{m})^2}}+\frac{\left|\frac{-1}{m}\right|}{\sqrt{3}\sqrt{1+(\frac{-1}{m})^2}}\right)\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}, & 0\leq\left|\frac{-1}{m}\right|\leq\sqrt{3}\\ \frac{2\left|\frac{-1}{m}\right|}{\sqrt{3}\sqrt{1+(\frac{-1}{m})^2}}, & \sqrt{3}\leq\left|\frac{-1}{m}\right|\\ \frac{2}{\sqrt{3}}, & \left|\frac{-1}{m}\right|\to\infty\end{cases}.$$

Theorem 2. *Let the base and height of a triangle in the plane $\mathbb{R}^2_{\pi 3}$ be $c_1$and h_1 . Let the base and length of the same triangle in $\mathbb{R}^2$ be c and h. Let the slope of the base of the triangle in $\mathbb{R}^2$ be m, then the area of the triangle in the plane $\mathbb{R}^2_{\pi 3}$ is*

$$S=\begin{cases}\frac{\sqrt{3}}{4}c_1h_1, & m=0\\ \frac{3\sqrt{1+m^2}}{4(\sqrt{3}|m|+1)}c_1h_1, & m=\infty\\ \frac{3(1+m^2)}{4(\sqrt{3}+|m|)}c_1h_1, & 0<m\leq\frac{1}{\sqrt{3}}\\ \frac{3(1+m^2)}{2(\sqrt{3}+|m|)(\sqrt{3}|m|+1)}c_1h_1, & \frac{1}{\sqrt{3}}<|m|\leq\sqrt{3}\\ \frac{3(1+m^2)}{4|m|(\sqrt{3}|m|+1)}c_1h_1, & \sqrt{3}\leq|m|, m\neq\infty.\end{cases}$$

Proof. According to the position of m the following eight main cases are possible:

Case 1) If the base of a triangle in the plane $\mathbb{R}^2_{\pi 3}$ is parallel to the $x-$axis, that is, $m=0$, the slope of the height is ∞. In this case;

$$S=\frac{1}{2}ch=\frac{\sqrt{3}}{4}c_1h_1$$

© 2022 BISKA Bilisim Technology

is found.

Case 2) If the base of a triangle in the plane $\mathbb{R}^2_{\pi 3}$ is parallel to the $y-$axis, that is, $m=\infty$, the slope of the height is 0. In this case;

$$c_1=\frac{2}{\sqrt{3}}c \text{ and } h_1=\left(\frac{1}{\sqrt{1+(\frac{-1}{m})^2}}+\frac{\left|\frac{-1}{m}\right|}{\sqrt{3}\sqrt{1+(\frac{-1}{m})^2}}\right)h,$$

$$\begin{aligned} S&=\frac{1}{2}ch \\ &=\frac{3\sqrt{1+m^2}}{4(\sqrt{3}\,|m|+1)}c_1h_1 \end{aligned}$$

is found.

Case 3) If the slope of the base of a triangle in the plane $\mathbb{R}^2_{\pi 3}$ is $0<m\leq\frac{1}{\sqrt{3}}$ (the angle θ between the base and the $x-$axis is $0<\theta\leq\frac{\pi}{6}$ and the angle θ between the height and the $x-$axis is $\frac{\pi}{2}<\theta\leq\frac{2\pi}{3}$) in this case;

$$c_1=\left(\frac{1}{\sqrt{1+m^2}}+\frac{|m|}{\sqrt{3}\sqrt{1+m^2}}\right)c \text{ and } h_1=\left(\frac{1}{\sqrt{1+(\frac{-1}{m})^2}}+\frac{\left|\frac{-1}{m}\right|}{\sqrt{3}\sqrt{1+(\frac{-1}{m})^2}}\right)h,$$

$$\begin{aligned} S&=\frac{1}{2}ch \\ &=\frac{3(1+m^2)}{4\,|m|\,(\sqrt{3}\,|m|+1)}c_1h_1, \end{aligned}$$

is found.

Case 4) If the slope of the base of a triangle in the plane $\mathbb{R}^2_{\pi 3}$ is $\frac{1}{\sqrt{3}}<m\leq\sqrt{3}$ (the angle θ between the base and the $x-$axis is $\frac{\pi}{6}<\theta\leq\frac{\pi}{3}$ and the angle θ between the height and the $x-$axis is $\frac{2\pi}{3}<\theta\leq\frac{5\pi}{6}$) in this case;

$$c_1=\left(\frac{1}{\sqrt{1+m^2}}+\frac{|m|}{\sqrt{3}\sqrt{1+m^2}}\right)c \text{ and } h_1=\left(\frac{2\left|\frac{-1}{m}\right|}{\sqrt{3}\sqrt{1+(\frac{-1}{m})^2}}\right)h,$$

$$\begin{aligned} S&=\frac{1}{2}ch \\ &=\frac{3(1+m^2)}{4(\sqrt{3}+|m|)}c_1h_1 \end{aligned}$$

is found.

Case 5) If the slope of the base of a triangle in the plane $\mathbb{R}^2_{\pi 3}$ is $\sqrt{3}<m<\infty$ (the angle θ between the base and the $x-$axis is $\frac{\pi}{3}<\theta\leq\frac{\pi}{2}$ and the angle θ between the height and the $x-$axis is $\frac{5\pi}{6}<\theta\leq\pi$) in this case;

$$c_1=\left(\frac{2\,|m|}{\sqrt{3}\sqrt{1+m^2}}\right)c \text{ and } h_1=\left(\frac{1}{\sqrt{1+(\frac{-1}{m})^2}}+\frac{\left|\frac{-1}{m}\right|}{\sqrt{3}\sqrt{1+(\frac{-1}{m})^2}}\right)h,$$

© 2022 BISKA Bilisim Technology

$$S = \frac{1}{2}ch$$
$$= \frac{3(1+m^2)}{4\left|m\right|(\sqrt{3}\left|m\right|+1)}c_1 h_1$$

is found.

Case 6.) If the slope of the base of a triangle in the plane $\mathbb{R}^2_{\pi 3}$ is $\infty < m < -\sqrt{3}$ (the angle θ between the base and the $x-$axis is $\frac{\pi}{2} < \theta \leq \frac{2\pi}{3}$ and the angle.θ between the height and the $x-$axis is $\pi < \theta \leq \frac{7\pi}{6}$) in this case;

$$c_1 = \left(\frac{2\left|m\right|}{\sqrt{3}\sqrt{1+m^2}}\right)c \text{ and } h_1 = \left(\frac{1}{\sqrt{1+(\frac{-1}{m})^2}} + \frac{\left|\frac{-1}{m}\right|}{\sqrt{3}\sqrt{1+(\frac{-1}{m})^2}}\right)h,$$

$$S = \frac{1}{2}ch$$
$$= \frac{3(1+m^2)}{4\left|m\right|(\sqrt{3}\left|m\right|+1)}c_1 h_1$$

is found.

Case 7.) If the slope of the base of a triangle in the plane $\mathbb{R}^2_{\pi 3}$ is $-\sqrt{3} < m < \frac{-1}{\sqrt{3}}$ (the angle θ between the base and the $x-$axis is $\frac{2\pi}{3} < \theta \leq \frac{5\pi}{6}$ and the angle θ between the height and the $x-$axis is $\frac{7\pi}{6} < \theta \leq \frac{4\pi}{3}$) in this case;

$$c_1 = \left(\frac{\sqrt{3}+\left|m\right|}{\sqrt{3}\sqrt{1+m^2}}\right)c \text{ and } h_1 = \left(\frac{1}{\sqrt{1+(\frac{-1}{m})^2}} + \frac{\left|\frac{-1}{m}\right|}{\sqrt{3}\sqrt{1+(\frac{-1}{m})^2}}\right)h,$$

$$S = \frac{1}{2}ch$$
$$= \frac{3(1+m^2)}{2(\sqrt{3}+\left|m\right|)(\sqrt{3}\left|m\right|+1)}c_1 h_1$$

is found.

Case 8.) If the slope of the base of a triangle in the plane $\mathbb{R}^2_{\pi 3}$ is $-\frac{-1}{\sqrt{3}} < m < 0$ (the angle θ between the base and the $x-$axis is $\frac{5\pi}{6} < \theta \leq \pi$ and the angle θ between the height and the $x-$axis is $\frac{4\pi}{3} < \theta \leq \frac{3\pi}{2}$), in this case;

$$c_1 = \left(\frac{1}{\sqrt{1+m^2}} + \frac{\left|m\right|}{\sqrt{3}\sqrt{1+m^2}}\right)c \text{ and } h_1 = \left(\frac{2\left|\frac{-1}{m}\right|}{\sqrt{3}\sqrt{1+(\frac{-1}{m})^2}}\right)h,$$

$$S = \frac{1}{2}ch$$
$$= \frac{3(1+m^2)}{4(\sqrt{3}+\left|m\right|)}c_1 h_1$$

is found.

© 2022 BISKA Bilisim Technology

References

[1] Z. Akça, S. Nazlı,.On the versions in the plane $\mathbb{R}^2_{\pi 3}$of some Euclidean theorems, New Trends in Mathematical Sciences, Vol.10, (2022),.20-27.

[2] Z. Akça, S. Nazlı, On The Trigonometric Functions in $\mathbb{R}^2_{\pi 3}$, Konuralp Journal of Mathematics, Vol. 8, (2020), 429-437.

[3] A. Bayar, R. Kaya, On Isometries of $\mathbb{R}^2_{\pi n}$, Hacettepe Journal of Mathematics and Statistics, Vol. 40 (5) (2011), 673-679.

[4] A. Bayar, S. Ekmekçi, and M. Özcan, "On Trigonometric Functions and Cosine and Sine Rules in Taxicab Plane," International Electronic Journal Of Geometry, vol. 2, no. 1, pp. 17–24, 2009.

[5] I. Kocayusufoglu, T. Ada, On The Iso-taxicab Trigonometry, Applied Sciences, Vol. 8 (2006), 101-111.

[6] I. Kocayusufoglu, Trigonometry on Iso-taxicab Geometry, Mathematical and Computational Applications, Vol. 5, No. 3 (2002),.201-212.

[7] K. O. Sowell, Taxicab Geometry-a New Slant, Mathematics Magazine, Vol. 62, (1989), 238-248.

[8] M. Özcan, S. Ekmekçi, and A. Bayar, "The Taxicab Lengths under Rotations," The Pi Mu Epsilon Journal, vol. 11, no. 7, pp. 381–384, 2002.

© 2022 BISKA Bilisim Technology