New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2022.496

On Embedding the Projective Plane $PG(2,4)$ to the Projective Space $P(4,4)$

Ayşe Bayar, Ziya Akça and Süheyla Ekmekçi

Eskişehir Osmangazi University, Department of Mathematics and Computer Science, 26480 Eskişehir, Turkey.

Received: 8 November 2022 , Accepted: 27 December 2022
Published online: 29 December 2022.

Abstract: In this study, all embeddings that map each line of projective plane order 4 to an oval of Projective Space of dimensional 4 will be investigated and it was shown that the image of these maps generate projective spaces P G(4, 4).

Keywords: Projective Space; Subgeometry; Embedding, Conic Plane.

1 Introduction and Preliminaries

Hall showed that an approximate linear space containing 4 points, three of which are not collinear, can be embedded in a projective plane in 1943 [3]. In 1963 Bruck gave the relations between the order of a finite projective plane and the orders of its sub-planes [2]. After the publication of Hirschfeld and Thas's book General Galois Geometries [4], Thas and Maldeghem gave embeddings for projective spaces derived from prime-order field. Embeddings were given with the help of various transformations depending on the order between the projective spaces $[1,\ 2,\ 9,10,11]$.

In Thas and Maldeghem, the classification all embeddings $\theta : PG(n,q) \to PG(d,q)$, with $d \geq \frac{n(n+3)}{2}$, such that θ maps the set of points of each line to a set of coplanar points and such that the image of θ generates $PG(d,q)$ is presented [9]. They give a common characterization of quadric Veroneseans and union of projections. Then they introduce the notation of a generalized Veronesean embedding by this characterization.

In Akça et.al. (2012), the classification all embeddings $\theta : PG(n,K) \to PG(d,F)$, with $d \geq \frac{n(n+3)}{2}$ and K and F skew field, such that θ maps the set $PG(d,F)$, and such that the image of θ generates $PG(d,F)$ was given [1].

A 4th order projective plane with 21 points and 21 lines has 5 points on each line. Also, on a non-singular conic in the same plane, there are 5 points, of which any three are non-collinear [7]. Therefore, the aim of the study is to investigate the existence of an embedding transformation from the 4th order projective plane to the 4-dimensional projective space with the help of ovals with 5 points, three of which are non-collinear, in the 4-dimensional projective space, which has 341 points and 341 hyperplanes. Linear embeddings can be defined using the projection transform into 4-dimensional projective space. The main purpose of this study is to investigate whether it is possible to embed the non-linear lines of the projective plane with the help of transformations that map the conics of the 4-dimensional projective space.

* Corresponding author e-mail: e-mail: zakca@ogu.edu.tr

1.1 Projective space

Definition 1. *([6]) An (axiomatic) projective plane P is an incidence structure $(N,D,\circ)$ with N a set of points, D a set of lines and $\circ$ an incidence relations, such that the following axioms are satisfied:*

(i) *every pair of distinct points are incident with a unique common line;*
(ii) *every pair of distinct lines are incident with a unique common point;*
(iii) *P contains a set of four points with the property that no three of them are incident with a common line.*

A closed configuration S of P is a subset of $N \cup D$ that is closed under taking intersection points of any pair of lines in S and lines spanned by any pair of distinct points of S. We denote the line in P spanned by the points p and q by $\langle p,q \rangle$.

Definition 2. *([4]) Let $V = V(n+1,K)$ be vector space with $n+1$ dimension on the field K. For an equivalence relation on the vectors of $V-\{0\}$, $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\} \in V-\{0\}$ and $\forall t \in K_0$, such that $X \sim Y \Leftrightarrow y_i = tx_i$, $i = 1, 2, ... n$, the equivalence classes on $V-\{0\}$ are one-dimensional subspaces formed by subtracting the zero vector from V. The set of these equivalence classes is called the n-dimensional projective space over the field K and it is indicated by $PG(n,K)$. If the qth order Galois field is taken as the field K, the projective space coordinated with the elements of this field is of order q. The obtained n-dimensional projective space is denoted by $PG(n,q)$.*

Theorem 1. *([4]) Let F be any field. A point-line geometry is a triple $(N,D,\circ)$ consisting of the points set N, the lines set D determined algebraically with the elements of the field and the incidence relation $\circ$. Obviously,*

$$N = \{(x_1,\, x_2,\, x_3) : x_i \in S, \exists\, x_i \neq 0,\ (x_1,\, x_2,\, x_3) \equiv \lambda\,(x_1,\, x_2,\, x_3),\ \lambda \in F\backslash\{0\}\}$$
$$D = \{[a_1,\, a_2,\, a_3] : a_i \in S,\ \exists a_i \neq 0,\ [a_1,\, a_2,\, a_3] \equiv \mu\,[a_1,\, a_2,\, a_3],\ \mu \in F\backslash\{0\}\}$$
$$\circ : (x_1,\, x_2,\, x_3) \circ [a_1,\, a_2,\, a_3] \Leftrightarrow a_1x_1 + a_2x_2 + a_3\,x_3 = 0.$$

Definition 3.*Any point in N is represented by a triple (x_1, x_2, x_3) where x_1, x_2, x_3 are not all zero. Nonzero multiples of a triple represent the same point. Similarly, any line in D is represented by a triple $[a_1, a_2, a_3]$ where a_1, a_2, a_3 are not all 0. This point-line geometry $(N,D,\circ)$ defined by F is a projective planene and is denoted by P_2F. Let r and p be a positive integer and a prime number, respectively. The projective plane of order $n = p^r$ over the finite Galois field $F = GF(p^r)$ of p^r elements by $P_2F = PG(2,p^r)$,* [3].

By expanding the field from a given field, new field are produced as follows:
Let F be any field, and $p(x)$ an irreducible polynomial of degree greater than 1 on this field. On the set

$$S = \{k(x) : k(x) \text{ is any polynomial of degree less than non the field } F\}$$

addition and multiplication operations are, in the known sense, addition and multiplication operations in polynomials. Considering elements of the field F,

$$k(x) \oplus k^{'}(x) = t(x),\ k(x) \odot k^{'}(x) = s(x)$$

is obtained. For this geometric structure, the Galois field $GF(2) = \{0,\ 1\}$ which has characteristic 2 and the $(S,\oplus,\odot)$ field, which is isomorphic to the $GF(2^2)$ body, with $f(t) = t^2+t+1$ being the irreducible polynomial on this body, are obtained as follows. The field $GF(2)$ is expanded to the set S with first-order polynomials such that λ is the root of the equation $f(\lambda) = \lambda^2+\lambda+1$ and $\lambda \notin GF(2)$. From the equality $\lambda^2+\lambda+1 = 0$, by using $\lambda^2 = \lambda+1$, the set $S =$

D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}	D_{14}	D_{15}	D_{16}	D_{17}	D_{18}	D_{19}	D_{20}
N_0	N_0	N_0	N_0	N_0	N_1	N_1	N_1	N_1	N_2	N_4	N_3	N_2	N_2	N_2	N_3	N_4	N_4	N_3	N_4	N_3
N_1	N_5	N_9	N_{10}	N_{11}	N_5	N_6	N_8	N_7	N_5	N_5	N_5	N_6	N_7	N_8	N_6	N_6	N_7	N_8	N_8	N_7
N_2	N_6	N_{14}	N_{12}	N_{13}	N_9	N_{14}	N_{13}	N_{14}	N_{14}	N_{11}	N_{12}	N_9	N_{11}	N_{10}	N_{11}	N_{10}	N_9	N_9	N_{11}	N_{10}
N_3	N_7	N_{15}	N_{17}	N_{18}	N_{10}	N_{17}	N_{16}	N_{19}	N_{19}	N_{15}	N_{16}	N_{12}	N_{16}	N_{15}	N_{15}	N_{16}	N_{18}	N_{17}	N_{12}	N_{13}
N_4	N_7	N_{16}	N_{19}	N_{20}	N_{11}	N_{18}	N_{19}	N_{20}	N_{20}	N_{17}	N_{18}	N_{13}	N_{17}	N_{18}	N_{19}	N_{20}	N_{19}	N_{20}	N_{14}	N_{14}

$\{a\lambda+b:\ a,\ b\in F,\ \lambda^2=\lambda+1\}$ is obtained as $\{0,\ 1,\ \lambda,\ \lambda+1\}$. On this set, if addition and multiplication operations are defined as

$$\oplus:(a_1\lambda+b_1)\oplus(a_2\lambda+b_2)=(a_1+a_2)\lambda+(b_1+b_2)$$
$$\odot:(a_1\lambda+b_1)\odot(a_2\lambda+b_2)=(a_1\lambda+b_1).(a_2\lambda+b_2),\ \lambda^2=\lambda+1.$$

Constructed $F=(S,\oplus,\odot)$ is a field that isomorphic to $GF(2^2)$. Points and lines of the projective plane defined with the help of the field $(S,\oplus,\odot)$ are

$$N=\{(x_1,\ x_2,\ x_3):x_i\in S,\ x_i\neq 0,\ (x_1,\ x_2,\ x_3)\equiv\lambda\,(x_1,\ x_2,\ x_3),\ \lambda\in F\setminus\{0\}\}$$
$$D=\{[a_1,\ a_2,\ a_3]:a_i\in S,\ \exists a_i\neq 0,\ [a_1,\ a_2,\ a_3]\equiv\mu\,[a_1,\ a_2,\ a_3],\ \mu\in F\setminus\{0\}\}$$
$$\circ:(x_1,\ x_2,\ x_3)\circ[a_1,\ a_2,\ a_3]\Leftrightarrow a_1x_1+a_2x_2+a_3\,x_3=0.$$

The point $(x_1,\ x_2,\ x_3)$ being on the line $[a_1,\ a_2,\ a_3]$ is defined by

$$(x_1,\ x_2,\ x_3)\odot[a_1,\ a_2,\ a_3]\Leftrightarrow a_1\odot x_1\oplus a_2\odot x_2\oplus a_3\odot x_3=0$$

Structure $(N,D,\circ)$ defined in this way is a projective plane with 21 points and 21 lines. The point set N of the projective plane of $PG(2,4)$ is $N=\{N_i|i=0,1,2,...,20\}$, where $N_0=(0,\ 1,\ 0)$, $N_1=(0,\ 0,\ 1)$, $N_2=(0,\ 1,\ 1)$, $N_3=\left(0,\ 1,\ t^2\right)$, $N_4=(0,\ 1,\ t)$, $N_5=(1,\ 1,\ 1)$, $N_6=(1,\ 0,\ 1)$, $N_7=(1,\ t,\ 1)$, $N_8=\left(1,\ t^2,\ 0\right)$, $N_9=(1,\ 1,\ 0)$, $N_{10}=\left(1,\ 1,\ t^2\right)$, $N_{11}=(1,\ 1,\ t)$, $N_{12}=\left(1,\ t,\ t^2\right)$, $N_{13}=\left(1,\ t^2,\ t\right)$, $N_{14}=(1,\ 0,\ 0)$, $N_{15}=(1,\ t,\ 0)$, $N_{16}=\left(1,\ t^2,\ 0\right)$, $N_{17}=\left(1,\ 0,\ t^2\right)$, $N_{18}=(1,\ 0,\ t)$, $N_{19}=\left(1,\ t^2,\ t^2\right)$, $N_{20}=(1,\ t,\ t)$.

The line set D of the projective plane of $PG(2,4)$ is $D=\{d_i|i=0,1,2,...,20\}$, where $D_0=[1,\ 0,\ 0]$, $D_1=[1,\ 0,\ 1]$, $D_2=[0,\ 0,\ 1]$, $D_3=[1,\ 0,\ t]$, $D_4=\left[1,\ 0,\ t^2\right]$, $D_5=[1,\ 1,\ 0]$, $D_6=[0,\ 1,\ 0]$, $D_7=[1,\ t,\ 0]$, $D_8=\left[1,\ t^2,\ 0\right]$, $D_9=[0,\ 1,\ 1]$, $D_{10}=\left[1,\ t^2,\ t\right]$, $D_{11}=\left[1,\ t,\ t^2\right]$, $D_{12}=[1,\ 1,\ 1]$, $D_{13}=[1,\ t,\ t]$, $D_{14}=\left[1,\ t^2,\ t^2\right]$, $D_{15}=\left[1,\ t^2,\ 1\right]$, $D_{16}=[1,\ t,\ 1]$, $D_{17}=\left[1,\ 1,\ t^2\right]$, $D_{18}=[1,\ 1,\ t]$, $D_{19}=\left[0,\ 1,\ t^2\right]$, $D_{20}=[0,\ 1,\ t]$. The incident relation table is given the following table:

1.2 Quadrics in Projective Plane PG(2,q)

A set of k points in $PG(2,q)$, of which any three are non-collinear, is called a k-arc such that q is a prime number and A conic in the projective plane $PG(2,q)$ is also a quadric. A conic in the qth order projective plane $PG(2,q)$ is the geometric locus of points defined as:

$$\left\{(x_0,x_1,x_2)\mid a_{00}x_0^2+a_{11}x_1^2+a_{22}x_2^2+a_{01}x_0x_1+a_{02}x_0x_2+a_{12}x_1x_2=0,a_{ij},x_i\in GF(q)\right\}$$

Since a scalar multiple of any solution of the above equation will also be a solution, the points of the $PG(2,q)$ projective plane are defined with this set. That is, a projective plane corresponds to a non-degenerate quadric equation, called a conic, containing exactly $q+1$ points. In a projective plane of order q, an oval is a set of $q+1$ points, of which any three are non-collinear. In $PG(2,q)$, the classic example of an oval is an irreducible conic. A linear equation and a non-degenerate quadric equation can have at most two common solutions. Storme ve Maldeghem [8] examined primitive arcs in $PG(2,q)$ projective plane. Beniamino Segre (1957) has shown that if the characteristic of the field is odd, then all ovals of $PG(2,q)$ are conics, but in even characteristic there are ovals which are not conics [7]. For any oval in a projective plane over a field of even characteristic, it can be shown that all of the tangent lines to the oval meet at a single point, called the knot of the oval . The knot of a conic is called the nucleus of the conic. In $PG(2,q)$ for q even , nucleus of the conic

$$a_{00}x_0^2+a_{11}x_1^2+a_{22}x_2^2+a_{01}x_0x_1+a_{02}x_0x_2+a_{12}x_1x_2=0$$

is the point $(a_{12},\ a_{02},\ a_{01})$. For q even, $(q+1)$ -arc can be uniquely completed to an oval by adding the nucleus.

The main purpose of this study is to investigate the feasibility of embedding the lines of the projective plane with the help of transformations mapping the cones of the 4-dimensional projective space. For this, first of all, the set of the points N, the set of the lines D and the structure of being on $\circ$ of a $P=(N,D,\circ)$ geometric structure isomorphic to the $PG(2,4)$ projective plane must be determined. It is well known that for a given field F, there is a projective plane whose points and lines can be co-ordinated algebraically with the elements of that field (Kaya, [6]).

2 The Embedding of PG(2,4) to PG(4,4)

In this study, we construct a transformation that is defined from the $PG(2,4)$ projective plane to the $PG(4,4)$ projective space by choosing a non-prime order and maps the set of points of each line of this projective plane to the set of points of an oval of the 4-dimensional projective space, it will be shown that the image set of the transformation produces a 4-dimensional projective space. In the previous section, a geometric structure $(N,D,\circ)$ isomorphic to the projective plane of the 4th order with 21 points and 21 lines was created. It is aimed to embed this structure in the projective space. In order for this geometric structure, whose points, lines and structure to be determined, can be embedded in the $PG(4,4)$ projective space with 341 points and 341 hyperplanes, we consider a set of points N' satisfying the conditions $N' \subseteq PG(4,4)$, $\left\langle N' \right\rangle = PG(4,4)$ and a set of planes D with each element being a subset of a conic plane in $PG(4,4)$ projective space.

Definition 4. *Let $S=(P,L)$ and $S'=(P',\ L')$ be near linear space. The transform f defined from S to S', which one-to-one maps S to the set of points P to the set of points of S' to P' and the set of lines of S to L to the set of lines of S', and preserves the incidence relation, is called an embedding transformation* [3].

Here, the existence of an embedding that maps the points N_i in the projective plane $PG(2,4)$ to the points N_i' in the projective space $PG(4,4)$ and the lines D_i in the projective plane $PG(2,4)$ to the planes D_i' in the projective space $PG(4,4)$ will be investigated, $i=0,1,\ldots,20$. We will use the conic equations to make an embedding from $PG(2,4)$ to $PG(4,4)$. We investigate whether the projective plane $PG(2,4)$ can be embedded in the projective space $PG(4,4)$ in two cases according to the nuclei of the two conic equations discussed. In the first case, two conics with the same nuclei will be selected, in the second case, two conics with different nuclei will be selected.

Theorem 2. *The projective plane $PG(2,4)$ cannot be embedded in the projective space $PG(4.4)$ if two cones with the same nuclei are used from the three conics spanning the projective space $PG(4.4)$.*

Proof: Let the nucleus of the conic $x_1^2=x_0x_2$ in the plane D_1' and the conic $x_1^2=x_0x_3$ in the plane D_5' be $(0,\ 1,\ 0,\ 0,\ 0)$.

We consider the triangle $N_0^{'}$, $N_1^{'}$, $N_5^{'}$ determined by the planes $D_0^{'}$, $D_1^{'}$, $D_5^{'}$in the projective space $PG(4,4)$. With a as a parameter, we can determine the following 11 points satisfying the conic equations above and spannig the projective space $PG(4,4)$. $N_0' = (0,\ 0,\ 1,\ 0,\ 0)$, $N_1' = (0,\ 0,\ 0,\ 1,\ 0)$, $N_2' = (0,\ 0,\ 0,\ 0,\ 1)$, $N_3' = (0,\ 0,\ a,\ 1,\ 1)$, $N_5' = (1,\ 0,\ 0,\ 0,\ 0)$, $N_6' = (1,\ 1,\ 1,\ 0,\ 0)$, $N_7' = \left(1,\ t,\ t^2,\ 0,\ 1\right)$, $N_8' = \left(1,\ t^2, t,\ 0,\ 0\right)$, $N_9' = (1,\ 1, 0,\ 1,\ 0)$, $N_{10}' = \left(1, t, 0, t^2, 0\right)$ and $N_{11}' = \left(1, t^2, 0, t, 0\right)$.

In this case, the other points of the geometric structure can be obtained as follows:

Considering that three points indicate a plane and the intersection of two planes is a projective point. Let $D_{12}^{'}$ be the plane spanned by points $N_2^{'}$, $N_6^{'}$, $N_9^{'}$ and $D_{20}^{'}$ be the plane spanned by points $N_3^{'}$, $N_7^{'}$, $N_{10}^{'}$. The intersection point of these two planes is found as $N_{13}^{'} = (0,\ 0,\ 1,\ 1,\ 0)$ by solving the plane equations. Let $D_0^{'}$ be the plane spanned by points $N_0^{'}$, $N_1^{'}$, $N_2^{'}$ and $D_{10}^{'}$ be the plane spanned by points $N_5^{'}$, $N_{13}^{'}$, $N_{15}^{'}$. The intersection point of these two planes is found as $N_4^{'} = (0,\ 0,\ 1,\ 1,\ 0)$. We seen that $N_4^{'} = (0,\ 0,\ 1,\ 1,\ 0) = N_{13}^{'}$ is obtained as indepedent of the parameter a. This is a contradiction.

Theorem 3. *The PG(2,4) projective plane can be embedded in the $PG(4.4)$ projective space if two cones with the different nuclei are used from the three conics spanning the projective space $PG(4.4)$.*

Proof. Let the nucleus of the conic $x_1^2 + x_0x_2 = 0$ in the plane $D_1^{'}$ be $M_1^{'} = (0,\ 1,\ 0,\ 0,\ 0)$ and let the nucleus of the conic $bx_4^2 + x_2x_3 + nx_2x_4 = 0$ in the plane $D_0^{'}$ be $M_0^{'} = (0,\ 0,\ 0,\ n,\ 1)$ and the equations of the plane $D_5^{'}$ be $x_1 = x_4,\ x_2 = 0$. We consider the triangle $N_0^{'}, N_1^{'}, N_5^{'}$ determined by the planes $D_0^{'}$, $D_1^{'}$ and $D_5^{'}$. Coordinates of these points are calculated as $N_0' = (0,\ 0,\ 1,\ 0,\ 0)$, $N_1' = (0,\ 0,\ 0,\ 1,\ 0)$ and $N_5' = (1,\ 0,\ 0,\ 0{,}0)$. The points $N_2' = (0,\ 0, b,\ 1+n,\ 1)$, $N_3' = \left(0,\ 0,\ b,\ n^2, nt\right)$, $N_4' = (0,\ 0,\ b,\ 0, n)$ can be choosen different from N_0', N_1' on D_0' and $N_6' = (1,\ 1{,}1,\ 0{,}0)$. Similarly, the points $N_7' = 1,\ t^2, t, 0, 0)$, $N_8' = \left(0,\ 0,\ b,\ n^2, nt\right)$ can be choosen different from N_0', N_5' on D_1' and the points $N_9' = (0,\ 1, 0, 0, 1)$, $N_{10}' = (a,\ 1, 0, 0, 1)$ can be choosen different from N_1', N_5' on D_5'.

Considering that three independent points indicate a plane and the intersection of two different planes is a projective point, the coordinates of the remaining ten points can be calculated according to the parameters $a,\ b,\ n$ in 27 different cases:

Case 1. $a = t^2,\ b = 1,\ n = t$

Let the conic plane $D_{12}^{'}$ be spanned by the points $N_2^{'}, N_6^{'}, N_9^{'}$ and denoted $\left\langle N_2^{'}, N_6^{'}, N_9^{'} \right\rangle$. The conic plane $D_{20}^{'}$ be spanned by the points $N_3^{'}, N_7^{'}, N_{10}^{'}$. The intersection point of these two planes is the point $N_{13}^{'}$. Every point on $D_{12}^{'}$ can be written as a linear sum of points $N_2^{'} = \left(0,0,1,t^2,1\right)$, $N_6^{'} = (1,1,1,0,0)$ and $N_9^{'} = (0,1,0,0,1)$. For any arbitrary point $(y_0, y_1, y_2, y_3, y_4)$ on $D_{12}^{'}$, the plane $D_{12}^{'}$ satisfies the equations $(y_2 - y_0)(1+t) = y_3$ and $(y_2 - y_0) + (y_1 - y_0) = y_4$ Since every point on $D_{20}^{'}$ is linear sum of $N_3^{'}$, $N_7^{'}$, $N_{10}^{'}$, the plane $D_{20chosenrime}$, this plane can be given by the equations $t^2 (y_0 + y_1 + y_2) = y_3, t^2 (y_0 + y_1 + y_2) = y_4$. It is easly seen that the intersection point of these two planes is the point $(1,0,t,t,t)$.

The plane $D_7^{'}$ denoted $\left\langle N_1^{'},\ N_8^{'},\ N_{13}^{'} \right\rangle$ is spanned by the points $N_1^{'}$, $N_8^{'}$, $N_{13}^{'}$ and this plane has the equations $t^2 (y_1 + y_4) = y_0$ and $t^2 y_1 + y_4 = y_2$. Similarly, the conic plane $D_{16}^{'}$ spanned with the points $N_4^{'}$, $N_6^{'}$, $N_{10}^{'}$ and has the equations $y_1 + ty_3 = y_0$ and $t(y_1 + y_2) + t^2 y_3 = y_4$. From the common solution of the equations of $D_7^{'}$ and $D_{16}^{'}$, the intersection point $N_{16}^{'}$ is found as $D_7^{'} \cap D_{16}^{'} = \left\langle N_1^{'},\ N_8^{'},\ N_{13}^{'} \right\rangle \cap \left\langle N_4^{'},\ N_6^{'},\ N_{10}^{'} \right\rangle = (0,\ t,\ 1,\ 1,\ t)$.

The remaining ten points can be calculated as

$$\begin{aligned}
N'_{11} &= D'_5 \cap D'_{13} = \left\langle N'_1, N'_5, N'_9 \right\rangle \cap \left\langle N'_2, N'_7, N'_{16} \right\rangle = (t,t,0,1,t)\,,\\
N'_{19} &= D'_7 \cap D'_{15} = \left\langle N'_1, N'_8, N'_{13} \right\rangle \cap \left\langle N'_3, N'_6, N'_{11} \right\rangle = \left(1,1,1,0,t^2\right),\\
N'_{12} &= D'_3 \cap D'_{11} = \left\langle N'_0, N'_{10}, N'_{19} \right\rangle \cap \left\langle N'_3, N'_5, N'_{16} \right\rangle = \left(1,t^2,t^2,1,0\right),\\
N'_{14} &= D'_9 \cap D'_{19} = \left\langle N'_2, N'_5, N'_{19} \right\rangle \cap \left\langle N'_4, N'_8, N'_{11} \right\rangle = (0,1,0,1,1)\,,\\
N'_{15} &= D'_2 \cap D'_{14} = \left\langle N'_0, N'_9, N'_{16} \right\rangle \cap \left\langle N'_2, N'_8, N'_{10} \right\rangle = \left(0,0,1,t^2,0\right),\\
N'_{17} &= D'_3 \cap D'_6 = \left\langle N'_0, N'_{10}, N'_{19} \right\rangle \cap \left\langle N'_1, N'_6, N'_{14} \right\rangle = \left(1,0,1,t^2,1\right),\\
N'_{18} &= D'_4 \cap D'_6 = \left\langle N'_0, N'_{10}, N'_{19} \right\rangle \cap \left\langle N'_1, N'_6, N'_{14} \right\rangle = \left(1,0,1,t^2,1\right),\\
N'_{20} &= D'_4 \cap D'_9 = \left\langle N'_0, N'_{11}, N'_{13} \right\rangle \cap \left\langle N'_2, N'_5, N'_{14} \right\rangle = \left(1,t^2,1,1,0\right),
\end{aligned}$$

respectively.

Let $N' = \{N'_i \,|i = 1,0,1,...,20\}$ be the point set and $D' = \{N'_i \,|i = 1,0,1,...,20\}$ be line set of the geometric sturucture $S = (N', D', \circ')$ such that the points of N' span $PG(4.4)$ and D'_i is a conic plane. Any two distinct elements N'_i and N'_j of the point set N' belong to unique member the line D'_k of the line set D' of S. Any two distinct elements of the line set D' have unique common point of the point set N'. S contains a set of four points of $N'$with the property that no three of them are incident with a common line of D'. So, The geometric sturucture $S = (N', D', \circ')$ is a projecxtive plane of order 4 such that S generates $PG(4,4)$ and is isomorfic to $PG(2,4)$. In this case, an embedding is obtained by mapping N_i points of the projective plane $PG(2,4)$ to the points N'_i of S in the projective space $PG(4.4)$ and mapping D_i lines of the projective plane $PG(2,4)$ to the conic planes D'_i of S the $PG(4.4)$ projective space.
Case 2. $a = 1,\ b = 1,\ n = 1$
In this case, let the conic plane D'_{12} be spanned by the points $N'_2,\ N'_6,\ N'_9$ and conic plane D'_{20} be spanned by the points $N'_3,\ N'_7,\ N'_{10}$. The intersection point of these two planes is the point N'_{13}. For any arbitrary point $(y_0,\ y_1,\ y_2,\ y_3,\ y_4)$ on D'_{12}, the plane D'_{12} has the equations $y_4 = y_1 + y_2,\ y_3 = 0$.. Since every point on D'_{20} is linear sum of $N'_3,\ N'_7,\ N'_{10}$, any arbitrary point $(y_0, y_1, y_2, y_3, y_4)$ on the plane D'_{20} satisfies the equations $y_1 = t^2\,(y_2 + y_3) + t y_0$ and $y_4 = t^2 y_2 + y_2 + t y_0$. It is seen that the intersection point of these two planes is the point $N'_{13} = (1,\ t,\ 0,\ 0,\ t)$.

Similarly, the plane $D'_7 = \left\langle N'_1,\ N'_8,\ N'_{13} \right\rangle$ is defined by the set

$$\left\{(y_0, y_1, y_2, y_3, y_4)\,|y_0 = t y_2 + t^2 y_4 y_1 = t^2 y_2 + y_4, y_i \in GF(2^2), i = 0,1,...,4\right\}$$

and the conic plane $D'_{16} = \left\langle N'_4,\ N'_6,\ N'_{13} \right\rangle$ is the set

$$\left\{(y_0,\ y_1,\ y_2,\ y_3,\ y_4)\,|y_4 = y_2 + y_1 y_0 = y_1, y_i \in GF(2^2), i = 0,1,...,4\right\}.$$

From the common solution of the equations of D'_7 and D'_{16}, the intersection point N'_{16} is not found.

An embedding cannot be obtained because no projective point other than the found points satisfying these equations can be found.
Case 3. $a = 1,\ b = 1,\ n = t$
When $a = 1,\ b = 1,\ n = t$, the points N'_{14} and N'_{11} have the coordinate $\left(t^2,\ t,\ 0,\ 1,\ t\right)$. An embedding cannot be obtained because these points coincide.

Case 4. $a = 1,\ b = 1,\ n = t^2$
In this case, since the points $N_{13}^{'} = (t^2,\ t,\ 0,\ 1,\ t)$ and $N_2^{'} = (t^2,\ t,\ 0,\ 1,\ t)$ coincide, so an embedding cannot be obtained.
Case 5. $a = 1,\ b = t,\ n = 1$
From calculations, $N_{13}^{'} = (t^2,\ t,\ 0,\ 1,\ t) = N_{13}^{'}$ is obtained, so an embedding cannot be obtained.
Case 6. $a = 1,\ b = t,\ n = t$
The coordinate of the point $N_{16}^{'}$ is obtained as $(0,\ 0,\ 0,\ 0,\ 0)$. Since there cannot be a point in projective space with these coordinates, an embedding cannot be obtained for $a = 1,\ b = t,\ n = t$.
Case 7. $a = 1,\ b = t,\ n = t^2$
In this case, since the points $N_{13}^{'} = (t^2,\ t,\ 0,\ 1,\ t)$ and $N_2^{'} = (t^2,\ t,\ 0,\ 1,\ t)$ coincide, so an embedding cannot be obtained.
Case 8. $a = 1,\ b = t^2,\ n = 1$
Similarly, since $N_{16}^{'} = (1,\ 1,\ t^2,\ 0,\ t^2) = N_{13}^{'}$, an embedding cannot be obtained.
Case 9. $a = 1,\ b = t,\ n = t^2$
In this case, since the points $N_{13}^{'} = (1,\ 1,\ 1,\ 0,\ 0)$ and $N_6^{'} = (1,\ 1,\ 1,\ 0,\ 0)$ coincide, so an embedding cannot be obtained.
Case 10. $a = 1,\ b = t^2,\ n = t^2$
The coordinate of the point $N_{16}^{'}$ is obtained as $(0,\ 0,\ 0,\ 0,\ 0)$. Since there cannot be a point in projective space with these coordinates, an embedding cannot be obtained for $a = 1,\ b = t^2,\ n = t^2$.
Case 11. $a = t,\ b = 1,\ n = 1$
In this case, since the points $N_{16}^{'} = (t,\ 1,\ 0,\ 1,\ 1)$ and $N_{10}^{'} = (t,\ 1,\ 0,\ 1,\ 1)$ coincide, so an embedding cannot be obtained.
Case 12. $a = t,\ b = 1,\ n = t$
In this case, since the points $N_{11}^{'} = (0,\ 1,\ 0,\ 0,\ 1)$ and $N_9^{'} = (0,\ 1,\ 0,\ 0,\ 1)$ coincide, so an embedding cannot be obtained.
Case 13. $a = t,\ b = 1,\ n = t^2$
In this case, since the points $N_{13}^{'} = (0,\ 0,\ 1,\ t,\ 1)$ and $N_2^{'} = (0,\ 0,\ 1,\ t,\ 1)$ coincide, so an embedding cannot be obtained.
Case 14. $a = t,\ b = t,\ n = 1$
In this case, since the points $N_{19}^{'} = (1,\ t^2,\ 1,\ 0,\ t)$ and $N_{13}^{'} = (1,\ t^2,\ 1,\ 0,\ t)$ coincide, so an embedding cannot be obtained.
Case 15. $a = t,\ b = t,\ n = t$
The coordinate of the point $N_{12}^{'}$ is obtained as $(0,\ 0,\ 0,\ 0,\ 0)$. Since there cannot be a point in projective space with these coordinates, an embedding cannot be obtained for $a = t,\ b = t,\ n = t$.
Case 16. $a = t,\ b = t,\ n = t^2$
In this case, since the points $N_{13}^{'} = (0,\ 0,\ 1,\ 1,\ t^2)$ and $N_3^{'} = (0,\ 0,\ 1,\ 1,\ t^2)$ coincide, so an embedding cannot be obtained.
Case 17. $a = t,\ b = t^2,\ n = 1$
In this case, since the points $N_{14}^{'} = (0,\ 0,\ 1,\ 0,\ t)$ and $N_4^{'} = (0,\ 0,\ 1,\ 0,\ t)$ coincide, so an embedding cannot be obtained.
Case 18. $a = t,\ b = t^2,\ n = t$
In this case, since the points $N_{19}^{'} = (1,\ t,\ t^2,\ 0,\ 0)$ and $N_8^{'} = (1,\ t,\ t^2,\ 0,\ 0)$ coincide, so an embedding cannot be obtained.
Case 19. $a = t,\ b = t^2,\ n = t^2$
In this case, since the points $N_{13}^{'} = (0,\ 0,\ 1,\ t^2,\ t)$ and $N_3^{'} = (0,\ 0,\ 1,\ t^2,\ t)$ coincide, so an embedding cannot be obtained.

Case 20. $a = t^2,\ b = 1,\ n = 1$
When the method of determining the coordinates of the points we applied in Case 1 above is applied, the coordinate of the point N'_{13} is obtained as $(0,\ 1,\ 0,\ 0,\ 1)$. Since this point coincide with the point N'_9, an embedding cannot be obtained for $a = t^2,\ b = 1,\ n = 1$. *Case 21.* $a = t^2,\ b = 1,\ n = t^2$
In this case, since the points $N'_{11} = (0,\ 1,\ 0,\ 0,\ 1)$ and $N'_9 = (0,\ 1,\ 0,\ 0,\ 1)$ coincide, so an embedding cannot be obtained.
Case 22. $a = t^2,\ b = t,\ n = 1$
In this case, since the points $N'_{13} = \left(1,\ t^2,\ t,\ 0,\ 0\right)$ and $N'_7 = \left(1,\ t^2,\ t,\ 0,\ 0\right)$ coincide, so an embedding cannot be obtained.
Case 23. $a = t^2,\ b = t,\ n = t$
In this case, since the points $N'_{11} = (0,\ 0,\ 1,\ 0,\ 1)$ and $N'_4 = (0,\ 0,\ 1,\ 0,\ 1)$ coincide, so an embedding cannot be obtained.
Case 24. $a = t^2,\ b = t,\ n = t^2$
In this case, since the points $N'_{13} = \left(0,\ 0,\ 1,\ 1,\ t^2\right)$ and $N'_2 = \left(0,\ 0,\ 1,\ 1,\ t^2\right)$ coincide, so an embedding cannot be obtained.
Case 25. $a = t^2,\ b = t^2,\ n = 1$
In this case, since the points $N'_{14} = (0,\ 0,\ 1,\ 0,\ t)$ and $N'_4 = (0,\ 0,\ 1,\ 0,\ t)$ coincide, so an embedding cannot be obtained.
Case 26. $a = t^2,\ b = t^2,\ n = t$
In this case, since the points $N'_{15} = (0,\ 0,\ 1,\ 1,\ t)$ and $N'_2 = (0,\ 0,\ 1,\ 1,\ t)$ coincide, so an embedding cannot be obtained.
Case 27. $a = t^2,\ b = t^2,\ n = t^2$
In this case, since the points $N'_{15} = \left(0,\ 0,\ 1,\ t^2,\ t\right)$ and $N'_2 = \left(0,\ 0,\ 1,\ t^2,\ t\right)$ coincide, so an embedding cannot be obtained.

3 Conclusion

The projective plane $PG(2,4)$ of order 4 with 21 points and 21 lines is aimed to embed the projective space $PG(4,4)$ with 341 points and 341 hyperplanes. We consider the point set N' satisfying the conditions $N' \subseteq PG(4,4)$, $\left\langle N' \right\rangle = PG(4,4)$ and the line set D' with each element being a conic plane in the projective space $PG(4,4)$.

The presence of embedding that maps the 21 line of the PG(2,4) projective plane to the 21 conic plane of the PG(4, 4) projective space was investigated. It has been shown that the projective plane $PG(2,4)$ cannot be embedded in the projective space $PG(4.4)$ if two cones with the same nuclei are used from the three conics spanning the projective space $PG(4.4)$. In the case of using two cones with different nuclei from the three cones spanning the projective space $PG(4.4)$, an embedding occurs in the projective space $PG(4.4)$ of the projective space $PG(2,4)$ only for the case $a = t^2,\ b = 1,\ n = t$, and embedding does not occur for the other 26 cases have shown.

This study will be an important resource for the studies to be done for embedding higher order projective planes in same order projective spaces.

Acknowledgment

This work was supported by the Scientific Research Projects Commission of Eskişehir Osmangazi University under Project Number 2016-1052.

References

[1] Z. Akça, A. Bayar, S. Ekmekci, R. Kaya, Thas J. A. and & H. Van Maldeghem, 2012. Generalized Veronesean embeddings of projective spaces, Part II. The lax case. Ars Combinatoria, CIII, pp. 65-80.

[2] H. R. Bruck, 1963, Finite nets, II. Uniqueness and imbedding, Pacif. J. Math. 13, 421-457.

[3] M. Hall, 1943, Projective planes, Trans. Am. Math. Soc. 54, 229-277.

[4] J.W.P. Hirschfeld and J. A. Thas, 1991, General Galois geometries, The Charendon Press, Oxford.

[5] J.W.P. Hirschfeld and J. A. Thas, 2016, General Galois Geometries, Springer Monographs in Mathematics, Springer-Verlag London, 143-195.

[6] R. Kaya, 2005. Projektif Geometri, Osmangazi Üniversitesi Yayınları, Eskişehir.

[7] B. Segre, 1955, Ovals in a finite projective plane. Canad. J. Math., 7, 414-416.

[8] L. Storme H. Van Maldeghem, Primitive Arcs in PG(2,q), J. Combin. Theory Ser. (A) 69 (1995), 200-216.

[9] J. A. Thas & H. Van Maldeghem, 2011. Generalized Veronesean embeddings of projective spaces. Combinatorica, 31 (5), 615-629.

[10] J. A. Thas & H. Van Maldeghem, Flat Lax and Weak Lax Embeddings of Finite Generalized Hexagons, European J. Combin. 19 (1998), 733-751.

[11] J. A. Thas & H. Van Maldeghem, 2004. Classification of Finite Veronesean Caps. European Journal of Combinatorics, 25, 275-285.