Communication in Mathematical Modeling and Applications
http://ntmsci.com/cmma

He's variational iteration method for solving linear and non-linear heat equations

Serdal Pamuk

Department of Mathematics, University of Kocaeli, Umuttepe Campus, 41380, Kocaeli - TURKEY

Received: 6 December 2022, Accepted: 12 December 2022
Published online: 30 December 2022.

Abstract: In this paper He's variational iteration method is applied to solve linear and non-linear heat equations. This method is based on the use of a general Lagrange multiplier in the construction of correction functional for the equation. In comparison with existing techniques, this method is very powerful, and the solution procedure is very simple since it uses only the prescribed initial condition. Some analytical results are presented.

Keywords: Approximate analytical solution, non-linear heat equations, soliton-like, cubic nonlinearity, He's variational iteration method.

1 Introduction

Finding the particular analytical solutions that have a physical or biological interpretation for the nonlinear equations is of fundamental importance since nonlinear phenomena play a crucial role in applied mathematics and science.

The main aim of this paper is to present applications of He's variational iteration method (VIM) to linear and non-linear heat equations of the form given by Eq.(1).

VIM is based on the incorporation of a general Lagrange multiplier in the construction of correction functional for the equation. This method has been proposed by Ji-Huan He [16] and is thoroughly used by many researchers (see e.g., [3,4,9-12,17,20]) to handle linear and non-linear problems. The VIM is very powerful and easy since it uses only the prescribed initial condition and does not require a specific treatment.

We consider the heat equation

$$u_t = u_{xx} + \varepsilon f(u), \tag{1}$$

where f is a linear or non-linear function of u, and ε is some parameter. Here, the indices t and x denote derivatives with respect to these variables. Construction of particular analytical solutions for nonlinear equations of the form of Eq.(1) is an important problem. Especially, finding an analytical solution that has a biological interpretation is of fundamental importance. In contrast to simple diffusion ($\varepsilon = 0$ case), when reaction kinetics and diffusion are coupled, travelling waves of chemical concentration exist, can effect a biochemical change, very much faster than straight diffusional processes governed by Eq.(1) with $\varepsilon = 0$ [8]. This coupling gives rise to reaction diffusion equation of the form of Eq.(1), where u is concentration and the term $f(u)$ represents the kinetics [8]. For example, it is known that when

* Corresponding author e-mail: spamuk@kocaeli.edu.tr

$f(u) = u^3$, Eq.(1) is an heat equation with cubic nonlinearity that admits soliton-like solutions [7,14].

Recently, some new methods such as Lie symmetry reduction method [6], and antireduction method [7] which transforms the nonlinear partial differential equations (PDE) to a system of ordinary differential equations (ODE) have been introduced in the research literature to find particular analytical solutions to PDE. But, finding analytical solutions of most nonlinear PDE generally requires new methods.

The particular analytical solutions of the nonlinear reaction diffusion equations of the form

$$u_t = (A(u)u_x)_x + B(u)u_x + C(u),$$

where $A(u)$, $B(u)$ and $C(u)$ are specially chosen smooth functions are obtained in [5]. This equation usually arises in mathematical biology [8]. In fact, Eq.(1) is a particular case of the last equation.

2 Variational iteration method (VIM)

To illustrate the basic idea of VIM, we consider the following general nonlinear system:

$$Lu(x,t) + Nu(x,t) = g(x,t), \tag{2}$$

where L is a linear operator, and N is a nonlinear operator, and $g(x,t)$ is the source inhomogeneous term.

According to the variational iteration method [16,17], one can construct the following iteration formulation:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)(Lu_n(x,s) + N\tilde{u}_n(x,s) - g(x,s))ds, \tag{3}$$

where λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$.

It is obvious now that the main steps of variational iteration method require first the determination of the Lagrangian multiplier λ that will be identified optimally. Having determined the Lagrangian multiplier, the successive approximations $u_{n+1}, n = 0,1,2,\cdots$, of the solution u will be readily obtained upon using any selective function u_0 [16,17]. Consequently, the solution is obtained as the limit of the resulting successive approximations, i.e.,

$$u = \lim_{n\to\infty} u_n. \tag{4}$$

3 Applications of VIM

Example 1. In this example we solve Eq.(1) with $\varepsilon = 1, f(u) = u$. We take $u(x,0) = \sin(\pi x)$ as the initial condition. According to VIM described above, a correction functional for this problem can be constructed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)\left(\frac{\partial u_n(x,s)}{\partial s} - \tilde{u}_n(x,s) - \frac{\partial^2\tilde{u}_n(x,s)}{\partial x^2}\right)ds, \tag{5}$$

where $n = 0,1,2,\cdots$, and λ is a Lagrange multiplier, $\tilde{u}_n$ is a restricted variation, i.e., $\delta\tilde{u}_n = 0$. To find the optimal value of λ, we make Eq.(5) stationary with respect to u_n, and obtain

$$\frac{\partial\lambda(t,s)}{\partial s} = 0, \tag{6}$$

$$1+\lambda(t,s)|_{s=t}=0. \tag{7}$$

The Lagrange multiplier can be identified as $\lambda=-1$, submitting the result into Eq.(5) leads to the following iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t(-1)\left(\frac{\partial u_n(x,s)}{\partial s}-u_n(x,s)-\frac{\partial^2 u_n(x,s)}{\partial x^2}\right)ds, \tag{8}$$

where $n=0,1,2,\cdots$. Beginning with an initial approximation $u_0(x,t)=u_0(x,0)=\sin(\pi x)$, we obtain the following successive approximations:

$u_1(x,t)=\sin(\pi x)+(1-\pi^2)t\sin(\pi x)$,
$u_2(x,t)=\sin(\pi x)+(1-\pi^2)t\sin(\pi x)+\frac{1}{2!}(1-\pi^2)^2t^2\sin(\pi x)$,
$u_3(x,t)=\sin(\pi x)+(1-\pi^2)t\sin(\pi x)+\frac{1}{2!}(1-\pi^2)^2t^2\sin(\pi x)+\frac{1}{3!}(1-\pi^2)^3t^3\sin(\pi x)$,
$\vdots$
$u_n(x,t)=\sin(\pi x)+(1-\pi^2)t\sin(\pi x)+\frac{1}{2!}(1-\pi^2)^2t^2\sin(\pi x)+\cdots+$
$\frac{1}{n!}(1-\pi^2)^nt^n\sin(\pi x)$.

By the use of Eq.(4), the particular analytical solution to this problem becomes

$$u(x,t)=e^{(1-\pi^2)t}\sin(\pi x), \tag{9}$$

in the closed form.

Example 2. We now solve Eq.(1) with $\varepsilon=-2, f(u)=u^3$, and $u(x,0)=\frac{1+2x}{x^2+x+1}$. A correction functional for this problem can be constructed as follows:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda(t,s)\left(\frac{\partial u_n(x,s)}{\partial s}+2\tilde{u}_n^3(x,s)-\frac{\partial^2\tilde{u}_n(x,s)}{\partial x^2}\right)ds, \tag{10}$$

where $n=0,1,2,\cdots$, and λ is a Lagrange multiplier, $\tilde{u}_n$ is a restricted variation, i.e., $\delta\tilde{u}_n=0$. The optimal value of the Lagrange multiplier is calculated to be $\lambda=-1$ as done in above example (see Eqs.(6)-(7)). Submitting this λ into Eq. (10) leads to the following iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t(-1)\left(\frac{\partial u_n(x,s)}{\partial s}+2u_n^3(x,s)-\frac{\partial^2 u_n(x,s)}{\partial x^2}\right)ds, \tag{11}$$

where $n=0,1,2,\cdots$. Beginning with an initial approximation $u_0(x,t)=u_0(x,0)=\dfrac{1+2x}{x^2+x+1}$, we obtain the following successive approximations:

$$u_1(x,t)=\frac{1+2x}{x^2+x+1}-\frac{6(1+2x)}{(x^2+x+1)^2}t,$$
$$u_2(x,t)=\frac{1+2x}{x^2+x+1}-\frac{6(1+2x)}{(x^2+x+1)^2}t+\frac{36(1+2x)}{(x^2+x+1)^3}t^2,$$
$$u_3(x,t)=\frac{1+2x}{x^2+x+1}-\frac{6(1+2x)}{(x^2+x+1)^2}t+\frac{36(1+2x)}{(x^2+x+1)^3}t^2-\frac{216(1+2x)}{(x^2+x+1)^4}t^3,$$
$$\vdots$$
$$u_n(x,t)=\frac{1+2x}{x^2+x+1}-\frac{6(1+2x)}{(x^2+x+1)^2}t+\frac{36(1+2x)}{(x^2+x+1)^3}t^2-\cdots+\frac{(-1)^n6^n(1+2x)}{(x^2+x+1)^{n+1}}t^n.$$

The VIM admits the use of Eq.(4), therefore one obtains the particular analytical exact solution to this problem

$$u(x,t)=\frac{1+2x}{6t+x^2+x+1}, \tag{12}$$

in the closed form.

Example 3. In this example we solve Eq.(1) with $\varepsilon = -1, f(u) = 2u^3 + u^2$, and $u(x,0) = 1/x$. This time a correction functional for this problem can be constructed as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)\left(\frac{\partial u_n(x,s)}{\partial s} + 2\tilde{u}_n^3(x,s) + \tilde{u}_n^2(x,s) - \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2}\right) ds, \tag{13}$$

where $n = 0,1,2,\cdots$, and λ is a Lagrange multiplier, $\tilde{u}_n$ is a restricted variation, i.e., $\delta\tilde{u}_n = 0$. Again, the optimal value of the Lagrange multiplier for this problem is calculated to be $\lambda = -1$. Submitting this λ into Eq. (13) leads to the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (-1)\left(\frac{\partial u_n(x,s)}{\partial s} + 2u_n^3(x,s) + u_n^2(x,s) - \frac{\partial^2 u_n(x,s)}{\partial x^2}\right) ds, \tag{14}$$

where $n = 0,1,2,\cdots$. We begin with an initial approximation $u_0(x,t) = u_0(x,0) = \frac{1}{x}$, and obtain the following successive approximations:

$$\begin{aligned}
u_1(x,t) &= \frac{1}{x} - \frac{t}{x^2},\\
u_2(x,t) &= \frac{1}{x} - \frac{t}{x^2} + \frac{t^2}{x^3},\\
u_3(x,t) &= \frac{1}{x} - \frac{t}{x^2} + \frac{t^2}{x^3} - \frac{t^3}{x^4},\\
&\vdots\\
u_n(x,t) &= \frac{1}{x} - \frac{t}{x^2} + \frac{t^2}{x^3} - \cdots + (-1)^n \frac{t^n}{x^{n+1}}.
\end{aligned}$$

By the aid of Eq.(4) we obtain the particular analytical solution to this problem

$$u(x,t) = \frac{1}{x+t}, \tag{15}$$

in the closed form.

4 Conclusion and results

In this paper, we provide some applications of VIM to solve linear and non-linear heat equations of the form in Eq. (1). We see that the VIM uses only the prescribed initial conditions and does not require a specific treatment.

On the other hand, the first two problems studied in Example 1 and Example 2 have been solved in [14] using Adomian decomposition method (ADM) [1,2], which is widely used by many scientists, e.g., [13,15,18,19]. ADM is an iterative method which provides approximate analytical solutions in the form of an infinite power series for nonlinear equations.

Although VIM and ADM give the same results for both of the problems, the VIM needs not to calculate Adomian polynomials, and it is very straightforward, and the solution procedure is very simple, as stated in [16].

Also, in their calculations of the exact solutions of various kinds of heat-like and wave-like equations, the authors of Ref.[16] pointed out that contrary to Adomian method, VIM needs no calculation of Adomian polynomial, only simple operation is needed. Another nice comparison between ADM and VIM is given by Wazwaz [17]. In his study he concludes the following: VIM gives several successive approximations through using the iteration of the correction functional. However, ADM provides the components of the exact solution, where these components should follow the summation of an infinite power series. Moreover, the VIM requires the evaluation of the Lagrangian multiplier λ, whereas ADM requires the evaluation of the Adomian polynomials that mostly require tedious algebraic calculations. More importantly, the VIM reduces the volume of calculations by not requiring the Adomian polynomials, hence the iteration is direct and straightforward. However, ADM requires the use of Adomian polynomials for nonlinear terms, and this needs more work. For nonlinear equations that arise frequently to express nonlinear phenomenon, He's VIM facilitates the computational work and gives the solution rapidly if compared with ADM.

As a result, although the numerical results are almost the same, VIM is much easier, more convenient and efficient than ADM.

References

[1] Adomian G. Solving Frontier Problems of Physics: the Decomposition Method, Kluwer Academic Publishers, Boston, 1994.

[2] Adomian G. Solving Frontier Problems Modelled by Nonlinear Partial Differential Equations. Comput. Math. Appl 1991;22(8): 91-94.

[3] Assas LMB. Variational Iteration Method for Solving Coupled-KdV Equations, Chaos, Solitons & Fractals 2008;38: 1225-1228.

[4] Bildik N, Konuralp A. The use of Variational Iteration Method, Differential Transform Method and Adomian Decomposition Method for Solving Different Types of Nonlinear Partial Differential Equations, International Journal of Nonlinear Sciences and Numerical Simulation 2006;7(1):65-70.

[5] Cherniha RM. New Ans*ä*tze and Exact Solutions for Nonlinear Reaction - Diffusion Equations Arising in Mathematical Biology, Sym. Nonlinear Math. Phys 1997;1:138-146.

[6] Euler M, Euler N. Symmetries for a Class of Explicitly Space and Time Dependent (1+1)- Dimensional Wave Equations, Sym. Nonlinear Math. Phys 1997;1:70-78.

[7] Fushchych W, Zhdanov R. Antireduction and Exact Solutions of Nonlinear Heat Equations, Nonlinear Math. Phys 1994;1(1):60-64.

[8] Murray J.D. Mathematical Biology, Springer, Berlin, 1993.

[9] Noor MA, Noor KI, Mohyud-Din ST, Modified Variational Iteration Technique for Solving Singular Fourth-Order Parabolic Partial Differential Equations, Nonlinear Analysis: Theory, Methods & Applications, In Press, Corrected Proof, Available Online 11 November 2008.

[10] Noor MA, Mohyud-Din ST. Variational Iteration Method for Fifth-Order Boundary Value Problems Using He's Polynomials, Mathematical Problems in Engineering, Article ID 954794, (2008).

[11] Noor MA, Mohyud-Din ST. Variational Iteration Technique for Solving Higher Order Boundary Value Problems, Appl. Math. Comput 2007;189:1929-1942.

[12] Noor MA, Mohyud-Din ST. Variational Iteration Decomposition Method for Solving Higher Dimensional Initial Boundary Value Problems, International Journal of Nonlinear Science 2009;7(1): 39-49.

[13] S Pamuk, The Decomposition Method for Continuous Population Models for Single and Interacting Species, Appl.Math.Comput 163(2005)79-88.

[14] Pamuk S. An Application for Linear and Nonlinear Heat Equations by Adomian's Decomposition Method, Appl.Math.Comput 2005; 163:89-96.

[15] Pamuk S. Solution of the Porous Media Equation by Adomian's Decomposition Method, Phys.Lett. A 2005;344:184-188.

[16] Shou DH, He JH, Beyond Adomian Method: The Variational Iteration Method for Solving Heat-Like and Wave-Like Equations with Variable Coefficients, Phys.Lett. A 2008;372:233-237.

[17] Wazwaz AM, A Comparison Between the Variational Iteration Method and Adomian Decomposition Method, J. Comput. Appl.Math 2007;207:129-136.

[18] Wazwaz AM, Gorguis A. Exact Solutions for Heat-Like and Wave-Like Equations with Variable Coefficients, Appl.Math.Comput 2004;149(1):15-29.

[19] Wazwaz AM. A Reliable Technique for Solving the Wave Equation in an Infinite One-Dimensional Medium, Appl.Math.Comput 1998;92: 1-7.

[20] Yusufoglu E, Variational Iteration Method for Construction of Some Compact and Noncompact Structures of Klein-Gordon Equations, International Journal of Nonlinear Sciences and Numerical Simulation 2007;8(2):153-158.