New Trends in Mathematical Sciences

http://dx.doi.org/10.20852/ntmsci.2023.507

Bounds of Fibonacci-sum graph for some distance-based topological indices and an application on BFS algorithm

Gül Özkan Kızılırmak

Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey

Received: 5 May 2022, Accepted: 27 March 2023
Published online: 20 November 2023.

Abstract: The aim of this study is to obtain the lower and upper bounds for the distance-based Wiener index, Hyper-Wiener index and Harary index of the Fibonacci-sum graph. Firstly, the shortest distance of the vertices in the Fibonacci-sum graph is characterized. Afterwards, the numbers of decision vertices and leaves are obtained by applying the BFS algorithm to the Fibonacci-sum graph.

Keywords: BFS Algorithm, Eccentricity, Fibonacci-sum graph, Graph, Topological index

1 Introduction

Graph theory has many fields of study. Topological indices have been a common field of study for the chemists and mathematicians. The aim of the topological indices is to determine whether they are useful for determining chemical properties. We use in this study base on distance of two vertices as a topological index [1].

One of the indices used in this work is the Wiener index of a graph G defined as [2,3]

$$W(G) = \frac{1}{2}\sum_{u=1}^{n}\sum_{v=1}^{n} d_G(u,v).$$

Similarly, we use the Hyper-Wiener index of a graph G introduced as [4]

$$WW(G) = \frac{1}{2}\sum_{u,v\in V}(d_G(u,v) + d_G(u,v)^2).$$

The Harary index of a graph G is also used and it is defined as [5,6]

$$H(G) = \sum_{u,v\in V}\frac{1}{d_G(u,v)}.$$

* Corresponding author e-mail: gulozkan@gazi.edu.tr

Many special graphs have been introduced in the literature. One of them is the Fibonacci-sum graph. For each positive integer n, the Fibonacci-sum graph $G_n = (V,E)$ is defined [7] by two vertices forming an edge i.e.,

$$E = \{\{i, j\} : i, j \in V,\ i \neq j,\ i+j \text{ is a Fibonacci number}\}.$$

It is obvious that G_n is a simple graph. Now, we will present some results which will be used later.

Corollary 1. *For each $n \geq 1$, G_n is connected [8].*

Lemma 1. *Let $n \geq 2$ and k be so that $F_k \leq n < F_{k+1}$. Then in G_n, the vertex F_k has only neighbour, namely F_{k-1} [8].*

Corollary 2. *[8] Let $n \geq 1$ and let $k \geq 2$ be integers satisfying $F_k \leq n < F_{k+1}$. Then*

$$|E(G_n)| = \begin{cases} n + \frac{F_k+1}{2} - \frac{\left\lfloor \frac{4(k+1)}{3} \right\rfloor}{2}, & \text{if } n \leq \frac{F_{k+2}}{2}; \\ 2n + \frac{F_k+1}{2} - \frac{\left\lfloor \frac{4(k+1)}{3} \right\rfloor}{2} - \left\lceil \frac{F_{k+2}-1}{2} \right\rceil, & \text{if } n > \frac{F_{k+2}}{2}. \end{cases}$$

Inspired by the study [8], Lucas-sum graph was defined and some properties of this graph were obtained in [9]. After that, in [10] the spectral properties of Fibonacci-sum and Lucas-sum graphs were examined and some bounds were obtained. Also, in [11] another type of graph associated with Fibonacci numbers was studied. In the literature, there exist a lot of bound studies. Some of these studies include bounds for indices and some for eigenvalues [12,13].

On the other hand, graph algorithms have a special place in the field of artificial intelligence. Breadth First Search(BFS) is a recursive algorithm to search all the vertices of a tree or graph data structure [14]. BFS puts every vertex of the graph into two categories; visited and non-visited. It selects a single vertex in a graph and, after that, visits all the vertices adjacent to the selected node.

In this study, we first define eccentricity and the shortest distance between two vertices in the Fibonacci-sum graph. Then, we obtain some upper and lower bounds for the Wiener index, Hyper-Wiener index and Harary index of the Fibonacci-sum graph using distance. Finally, by using the BFS algorithm, we find the number of leaves and decision vertices.

2 Main Results

Definition 1. *Let F_k be the kth Fibonacci number. Then, the distance from u to the nearest Fibonacci number denoted by $\theta(u)$ is defined as follows:*

$$\theta(u) = \min_{u \in V} \left\{ d_{G_n}(u, v) \ : v = F_k \right\}.$$

We note that, if $u = F_r$, then $\theta(u) = 0$.

Now, we give the following definition, which characterize the shortest distance of the vertices in the Fibonacci-sum graph G_n.

Definition 2. *For $u, v \in V(G_n)$, we let F_k and F_t be the nearest Fibonacci numbers to u and v, respectively. Then, the shortest distance of the vertices u and v in the Fibonacci-sum graph G_n is*

$$d_{G_n}(u, v) = \theta(u) + \theta(v) + |k - t|.$$

Now, we will give upper and lower bounds for the Wiener index of the Fibonacci-sum graph using distance.

Theorem 1. *If G_n is Fibonacci-sum graph, then*

$$n^2 - n - m \leq W(G_n) \leq \binom{n}{2}(\varepsilon(1) + k - 2)$$

where n is the number of vertices, $m = |E(G_n)|$ is number of edge, and k is an integer satisfying $F_k \leq n$, $\varepsilon(1) = \max_{u \in V} \{d_G(u,1) \ : u \neq F_k\}$.

Proof. For $u,v \in V$, the number of $d_{G_n}(u,v)$ is $\binom{n}{2}$. Since $W(G_n) = \frac{1}{2}\sum_{u=1}^{n}\sum_{v=1}^{n} d_{G_n}(u,v)$,assuming all $d_{G_n}(u,v)$ to be 1, we get the lower bound for the Wiener index of G_n as

$$\binom{n}{2} \leq W(G_n).$$

Since the number of egdes with length 1 gives m, which is the number of egdes, the other lengths can be taken as 2 and the lower bound for $W(G_n)$ can be improved as

$$(\binom{n}{2} - m)2 + m.1 = n^2 - n - m \leq W(G_n).$$

On the other hand, the maximum of $d_{G_n}(u,v)$ is the sum of the maximum distance of F_k to 1 such that $F_k \leq n$ and the distance of 1 in $P: 1, F_2, F_3, ...F_k$ to a vertex v not on the P path. So,

$$max\ d_{G_n}(u,v) = \varepsilon(1) + |P(F_2,F_k)|.$$

Hence, by making the necessary calculations, we get the upper bound for the Wiener index of G_n as

$$W(G_n) \leq \binom{n}{2}(\varepsilon(1) + k - 2).$$

Next, we will give upper and lower bounds for the Hyper-Wiener index of the Fibonacci-sum graph using distance.

Theorem 2. *If G_n is a Fibonacci-sum graph then*

$$[3n(n-1)] - 4m \leq WW(G_n) \leq (\frac{n(n-1)}{2})(\varepsilon(1) + k - 2)(\varepsilon(1) + k - 1))$$

where n, m, k and $\varepsilon(1)$ are as in Theorem 1.

Proof. Since $WW(G_n) = \frac{1}{2}\sum_{u,v \in V}(d_{G_n}(u,v) + d_{G_n}(u,v)^2)$, assuming all $d_{G_n}(u,v)$ to be 1, we get the lower bound for the Hyper-Wiener index of G_n as

$$2\binom{n}{2} \leq WW(G_n).$$

In addition, there are $\binom{n}{2}$ elements in $\sum_{u,v \in V} d(u,v)$. If the length of m of these is 1 and the length of the others is 2, the lower bound for $WW(G_n)$ can be improved as

$$\begin{aligned}(6(\binom{n}{2} - m) + 2m) = &\ (6(\frac{n(n-1)}{2}) - 4m) \\ = &\ [3n(n-1)] - 4m \leq W(G_n).\end{aligned}$$

On the other hand, we have the upper bounds for the Hyper-Wiener index of G_n as

$$WW(G_n) \leq \frac{1}{2}(\varepsilon(1)+k-2)+(\varepsilon(1)+k-2)^2)$$

Hence, we get

$$WW(G_n) \leq (\frac{n(n-1)}{2})(\varepsilon(1)+k-2)(\varepsilon(1)+k-1)).$$

Now, we will give upper and lower bounds for the Harary index of the Fibonacci-sum graph using distance.

Theorem 3.*If G_n is Fibonacci-sum graph then*

$$\frac{n(n-1)}{2(\varepsilon(1)+k-2)} \leq H(G_n) \leq n^2-n-m$$

where n, m, k and $\varepsilon(1)$ are as in Theorem 1.

*Proof.*Since $d_{G_n}(u,v)$ must be minimum for $\frac{1}{d_{G_n}(u,v)}$ to be maximum and $d_{G_n}(u,v)$ must be maximum to be minimum, we have

$$\min\left\{\sum_{u,v\in V}\frac{1}{d_{G_n}(u,v)}\right\} \leq H(G_n) \leq \max\left\{\sum_{u,v\in V}\frac{1}{d_{G_n}(u,v)}\right\}.$$

Due to the structure of G_n, we get

$$\binom{n}{2}\frac{1}{(\varepsilon(1)+k-2)} \leq H(G_n) \leq (\binom{n}{2}-m)2+m.1$$

By making the necessary calculations, we have

$$\frac{n(n-1)}{2(\varepsilon(1)+k-2)} \leq H(G_n) \leq n(n-1)-m$$

For some values of n, Table 1 shows that the Wiener, Hyper-Wiener and Harary indices of the Fibonacci-sum graph are within the bounds that we found.

Table 1: The $W(G_n)$, $WW(G_n)$ and $H(G_n)$ indices of the Fibonacci-sum graph according to the values of n.

n	$W(G_n)$	$WW(G_n)$	$H(G_n)$
5	20	70	6.416
6	70	0.060 4	2.25
7	44	156	12.66
8	69	282	15.23
9	97	398	18.26

3 Application of Fibonacci-sum graph to BFS algorithm

Theorem 4.*In the spanning tree of a Fibonacci-sum graph with n points, the number of leaves is $n-F_{k-1}$ and the number of decision vertex is F_{k-1}, where F_{k-1} is the largest Fibonacci number such that $F_{k-1} \leq F_k \leq n$.*

*Proof.*If we create the BFS tree of an n-point Fibonacci-sum graph by applying the BFS algorithm starting from the highest grade point 2 to points 1-3..., this BFS tree is also a spanning tree of the Fibonacci-sum graph and has F_{k-1} decision vertices. Therefore, the number of leaves will be $n-F_{k-1}$.

Example 1. Consider a 12-point Fibonacci-sum graph. In this case, $F_k \leq n$, the larger F_k is 8, and F_{k-1} becomes 5. In other words, according to the statement of the theorem, the number of decision vertices is 5, the number of leaves is 7. If we start with the largest degree in the graph, 2, we first go to the vertices $1,3,6$ and 11. Then, if we go from vertex 3 to 5 and 10, from vertex 5 to 8, from vertex 1 to $4,7$ and 12 and from 4 to 9, the BFS tree is obtained. Here are the decision vertices; $1,2,3,4,5$ and leaves; $6,7,8,9,10,11$ and 12. Thus, it is seen that the number of decision vertices is 5 and the number of leaves is 7.

4 Conclusion

In this work, we introduced the shortest distance of the vertices in the Fibonacci-sum graph. Then, we obtained the lower and upper bounds for the distance-based Wiener index, Hyper-Wiener index and Harary index of the Fibonacci-sum graph. Also, we gave an application of BFS algoritm. Finally, we obtained the number of decision vertices and the number of leaves by applying the BFS algorithm to the Fibonacci-sum graph.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] R. C. Entringer, D.E. Jackson, D. A. Snyder, Distance in graphs, *Czech. Math. J.,* **26** (1976), 283-296.

[2] A.A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: Theory and applications, *Acta Appl. Math.,* **66** (2001), 211-249.

[3] H. Wiener, Structural determination of the paraffin boiling points, *J. Am. Chem. Soc.,* **69** (1947), 17-20.

[4] D.J. Klein, I. Lukovits, I. Gutman, On the definition of the hyper-Wiener index for cycle-containing structures, *J. Chem. Inf. Comput. Sci.,* **35** (1995), 50-52.

[5] Plavsic, S. Nikoli, N. Trinajstic, Z. Mihalic, On the Harary index for the characterization of chemical graphs, *J. Math.Chem.,* **12** (1993), 235-250.

[6] O. Ivanciuc, T. S. Balaban, A. T. Balaban, Design of topological indices. Part 4. Reciprocal distance matrix, related local vertex invariants and topological indices, *J. Math. Chem.,* **12** (1993), 309-318.

[7] K. Fox, W. B. Kinnersley, D. McDonald, N. Orlow, G. J. Puleo, Spanning paths in Fibonacci-Sum graphs, *Fibonacci Quart.,* **52** (2014), 46-49.

[8] A. Arman, S., D. S. Gunderson, P.C. Li, Properties of the Fibonacci-sum graph, arXiv:1710.10303v1[math.CO] 27 Oct 2017 https://arxiv.org/abs/1710.10303.

[9] D. Tasci, S., S. Büyükköse, G. Özkan Kizilirmak, E. Sevgi, Properties of Lucas-sum Graphs, *Journal of Science and Arts,* **51** (2020), 313-316.

[10] D. Tasci, S. G. Özkan Kizilirmak, E. Sevgi, S. Büyükköse, The Bounds for the Largest Eigenvalues of Fibonacci-sum and Lucas-sum Graphs, *TWMS J. App. Eng. Math,*

[11] A. Yurttas Gunes, S. Delen, M. Demirci, A.S. Cevik, I.N. Cangul, Fibonacci Graphs, *Symmetry,* **12** (2020), 21-28.

[12] S. Buyukkose, N. Mutlu The upper bound for the largest signless Laplacian eigenvalue Of weighted graphs, *Gazi University Journal of Science,* **28** (2015), 709-714.

[13] S. B. Nurkahli, S. Buyukkose, Wiener Index of Interval Weighted Graphs, *Journal of Science and Arts,* **1** (2021), 21-28.

[14] F. Xia, , K. Sun, S. Yu, A. Aziz, L. Wan, S. Pan, H. Liu, Graph Learning: A Survey," in IEEE Transactions on Artificial Intelligence, *IEEE Transactions on Artificial Intelligence,* **2** (2021), 109-127.