New Trends in Mathematical Sciences

http://dx.doi.org/10.20852/ntmsci.2023.508

An improvement of Hermite-Hadamard-Fejér inequality

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Received: 19 October 2022, Accepted: 27 March 2023
Published online: 20 November 2023.

Abstract: In this paper, we have obtained new Hermite Hadamard Fejér type inequality different from the classical fejér inequality. Thanks to this new inequality, new types of fractional integral inequalities obtained in recent years can be obtained in special cases.

Keywords: Fejer, Convex Functions, Fractional Integrals.

1 Introduction

The most important inequality in the theory of convex functions is Hermite-Hadamard's inequality in below [3,4]. If ξ is a convex function on $[\delta,\beta]$, then

$$\xi\left(\frac{\delta+\beta}{2}\right)\leq\frac{1}{\beta-\delta}\int\limits_{\delta}^{\beta}\xi(\chi)d\chi\leq\frac{\xi(\delta)+\xi(\beta)}{2}. \tag{1}$$

If ξ is concave on $[\delta,\beta]$, then the inequality (1) is reversed. It is worth noting that Hadamard's inequality can be seen as a refinement of the concept of convexity. In [2], Fejér gave weighted versions, the so-called Hermite Hadamard-Fejér inequality as the following:

Theorem 1.$\xi:[\delta,\beta]\to\mathbb{R}$, *be a convex function, the the inequality*

$$\xi\left(\frac{\delta+\beta}{2}\right)\int\limits_{\delta}^{\beta}w(\chi)d\chi\leq\frac{1}{\beta-\delta}\int\limits_{\delta}^{\beta}\xi(\chi)w(\chi)d\chi\leq\frac{\xi(\delta)+\xi(\beta)}{2}\int\limits_{\delta}^{\beta}w(\chi)d\chi \tag{2}$$

holds, where $w:[\delta,\beta]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $\chi=\frac{\delta+\beta}{2}$. When it is taken $w(\chi)=1$ to (2), (1) is obtained.

Definition 1.*Let $\xi\in L_1[\delta,\beta]$. The Riemann-Liouville integrals $J_{\delta+}^{\alpha}\xi$ and $J_{\beta-}^{\alpha}\xi$ of order $\alpha>0$ are defined by*

$$J_{\delta+}^{\alpha}\xi(\chi)=\frac{1}{\Gamma(\alpha)}\int\limits_{\delta}^{\chi}(\chi-\iota)^{\alpha-1}\xi(\iota)d\iota,\quad \chi>\delta,\quad J_{\beta-}^{\alpha}\xi(\chi)=\frac{1}{\Gamma(\alpha)}\int\limits_{\chi}^{\beta}(\iota-\chi)^{\alpha-1}\xi(\iota)d\iota,\quad \chi<\beta$$

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respectively. Here $\Gamma(\alpha)$ *is the Gamma function defined by* $\Gamma(\alpha)=\int_0^\infty e^{-\iota}\iota^{\alpha-1}d\iota$ *and* $J^0_{\delta+}\xi(\chi)=J^0_{\beta-}\xi(\chi)=\xi(\chi)$ *(see [9, page 69] and [16, page 4]). The beta function and incomplete beta function defined as follows:*

$$B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}=\int_0^1 \iota^{u-1}(1-\iota)^{v-1}d\iota,\quad u,v>0,$$

$$B_w(u,v)=\int_0^w \iota^{u-1}(1-\iota)^{v-1}d\iota \quad u,v>0 \text{ and } 0\leq w\leq 1.$$

In [10], Sarikaya et al. found out the Hermite-Hadamard inequalities in fractional integral as following:

Theorem 2.*Let* $\xi:[a,b]\to\mathbb{R}$ *be a positive function with* $0\leq\delta<\beta$ *and* $\xi\in L_1[\delta,\beta]$*. If* ξ *is a convex function on* $[\delta,\beta]$*, then the following inequalities for fractional integrals hold:*

$$\xi\left(\frac{\delta+\beta}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(\beta-\delta)^\alpha}\left[J^\alpha_{\delta+}\xi(\beta)+J^\alpha_{\beta-}\xi(\delta)\right]\leq\frac{\xi(\delta)+\xi(\beta)}{2} \tag{3}$$

with $\alpha>0$.

In [5], İşcan proved the following fractional Hermite-Hadamard-Fejér type inequality:

Theorem 3.*Let* $\xi:[\delta,\beta]\to\mathbb{R}$ *be a convex function with* $\delta<\beta$ *and* $\xi\in L[\delta,\beta]$*. If* $\omega:[\delta,\beta]\to\mathbb{R}$ *is nonnegative, integrable and symmetric to* $\frac{\delta+\beta}{2}$*, then the following inequality for fractional integrals holds:*

$$\xi\left(\frac{\delta+\beta}{2}\right)\left[J^\alpha_{\delta+}\omega(\beta)+J^\alpha_{\beta-}\omega(\delta)\right]\leq\left[J^\alpha_{\delta+}(\xi\omega)(\beta)+J^\alpha_{\beta-}(\xi\omega)(\delta)\right]\leq\frac{\xi(\delta)+\xi(\beta)}{2}\left[J^\alpha_{\delta+}\omega(\beta)+J^\alpha_{\beta-}\omega(\delta)\right] \tag{4}$$

with $\alpha>0$.

In [6,7,8], M Kunt et al. have seen that (3) inequality is the natural result of the (2) with respect to $\omega(x)=(\beta-x)^{\alpha-1}+(x-\delta)^{\alpha-1}$ in (2). A question that it is wondered that an inequality is the best from (3) inequality and is not the result of the (2) is brought to mind. In [8], Kunt et al. gave the answer of this question with the following theorem:

Theorem 4.*[8] Let* $\delta,\beta\in\mathbb{R}$ *with* $\delta<\beta$ *and* $\xi:[\delta,\beta]\to\mathbb{R}$ *be a convex function. If* $\xi\in L[\delta,\beta]$*, then the following inequality for fractional integral holds:*

$$\frac{\xi\left(\frac{\alpha\delta+\beta}{\alpha+1}\right)+\xi\left(\frac{\delta+\alpha\beta}{\alpha+1}\right)}{2}\leq\frac{\Gamma(\alpha+1)}{2(\beta-\delta)^\alpha}\left[J^\alpha_{\delta+}\xi(\beta)+J^\alpha_{\beta-}\xi(\delta)\right]\leq\frac{\xi(\delta)+\xi(\beta)}{2}$$

with $\alpha>0$.

In [1], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha>0$ as follows:

Definition 2.*Let* $\alpha\in(\varepsilon,\varepsilon+1]$ *and set* $\theta=\alpha-\varepsilon$ *then the left conformable fractional integral starting at* δ *if order* α *is defined by*

$$I^\delta_\alpha f(\iota)=\frac{1}{\varepsilon!}\int_\delta^\iota(\iota-\chi)^\varepsilon(\chi-\delta)^{\theta-1}\xi(\chi)d\chi,\ \iota>\delta$$

Analogously, the right conformable fractional integral is defined by

$${}^\beta I_\alpha\xi(\iota)=\frac{1}{\varepsilon!}\int_\iota^\beta(\chi-\iota)^\varepsilon(\beta-\chi)^{\theta-1}\xi(\chi)d\chi,\ \iota<\beta$$

Notice that if $\alpha=\varepsilon+1$ *then* $\theta=\alpha-\varepsilon=\varepsilon+1-\varepsilon=1$ *where* $\varepsilon=0,1,2,\ldots$ *and hence* $I^\delta_\alpha\xi(\iota)=I^\delta_{\varepsilon+1}\xi(\iota)$.

On recently years, conformable fractional integral is studied by author on the different of field. Also, in [11,12], Set et al. obtained both Hermite Hadamard and Hermite Hadamard Fejér type inequalities via conformable fractional integral as follow.

Theorem 5. *Let $\xi : [\delta,\beta] \to \mathbb{R}$ be a function with $0 \le \delta < \beta$ and $\xi \in L_1[\delta;\beta]$. If ξ is a convex function on $[\delta,\beta]$, then the following inequalities for conformable fractional integrals hold:*

$$\xi\left(\frac{\delta+\beta}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(\beta-\delta)^{\alpha}\Gamma(\alpha-n)}\left[I_{\alpha}^{\delta} f(\beta) + {}^{\beta}I_{\alpha}\xi(\delta)\right] \le \frac{\xi(\delta)+\xi(\beta)}{2} \tag{5}$$

with $\alpha \in (\varepsilon, \varepsilon+1]$.

Here,

Theorem 6. *Let $\xi : [\delta,\beta] \to \mathbb{R}$ be a convex function with $\delta < \beta$ and $f \in L[\delta,\beta]$. If $\omega : [\delta,\beta] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{\delta+\beta}{2}$, then the following inequality for fractional integrals holds:*

$$\xi\left(\frac{\delta+\beta}{2}\right)\left[I_{\alpha}^{\delta}\omega(\beta) + {}^{\beta}I_{\alpha}\omega(\delta)\right] \le \left[I_{\alpha}^{\delta}(\xi\omega)(\beta) + {}^{\beta}I_{\alpha}(\xi\omega)(\delta)\right] \tag{6}$$

$$\le \frac{\xi(\delta)+\xi(\beta)}{2}\left[I_{\alpha}^{\delta}\omega(\beta) + {}^{\beta}I_{\alpha}\omega(\delta)\right]$$

with $\alpha > 0$.

In [15], Turhan et al. have found out that (5) and (6) inequalities are the result of (2) respect to $w(\chi) = (\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon-1} + (x-\delta)^{\varepsilon}(\beta-\chi)^{\alpha-\varepsilon-1}$ and $w(\chi) = (\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon-1} + (\chi-\delta)^{\varepsilon}(\beta-\chi)^{\alpha-\varepsilon-1}g(\chi)$. And also Turhan et al. have revealed the following new inequality that is not the result of (2).

Theorem 7. *Let $\delta,\beta \in \mathbb{R}$ with $\delta < \beta$ and $\xi : [\delta,\beta] \to \mathbb{R}$ be a convex function. If $\xi \in L[\delta,\beta]$, then the following inequality for fractional integral holds:*

$$\frac{\xi\left(\frac{(\varepsilon+1)\delta+(\alpha-\varepsilon)\beta}{\alpha+1}\right) + \xi\left(\frac{(\alpha-\varepsilon)\delta+(\varepsilon+1)\beta}{\alpha+1}\right)}{2} \le \frac{\Gamma(\alpha+1)}{2(\beta-\delta)^{\alpha}\Gamma(\alpha-\varepsilon)}\left[I_{\alpha}^{\delta}\xi(\beta) + {}^{\beta}I_{\alpha}\xi(\delta)\right] \le \frac{\xi(\delta)+\xi(\beta)}{2} \tag{7}$$

with $\alpha > 0$.

In Literature, there are the many of Hermite Hadamard Fejér type inequalities. Hermite Hadamard type inequalities obtained via the Riemann- Liouvile fractional integral, conformable fractional integral and another several type fractional integrals are obtained with the help of the inequality (2). Aim of this paper, it is found out a new Hermite Hadamard Fejér type inequality that can be reduced to Hermite Hadamard inequalities for all fractional integrals (such as Riemann-Liouville and conformable fractional integrals).

2 Main Results

Theorem 8. *Let $\delta < \beta$ and $\xi : [\delta,\beta] \to \mathbb{R}$ be convex function. If $\omega : [\delta,\beta] \to \mathbb{R}$ is nonnegative, integrable, then the following inequalities hold*

$$\xi\left(\beta - \frac{\int_{\delta}^{\beta}\left(\int_{\delta}^{\chi}\omega(s)ds\right)d\chi}{\int_{\delta}^{\beta}\omega(s)ds}\right)\int_{\delta}^{\beta}\omega(\chi)d\chi \le \int_{\delta}^{\beta}\xi(\chi)\omega(\chi)d\chi \le \frac{\xi(\beta)}{\beta-\delta}\int_{\delta}^{\beta}(\chi-\delta)\omega(\chi)d\chi + \frac{\xi(\delta)}{\beta-\delta}\int_{\delta}^{\beta}(\beta-\chi)\omega(\chi)d\chi \tag{8}$$

and

$$\xi\left(\delta + \frac{\int_{\delta}^{\beta}\left(\int_{\chi}^{\beta}\omega(s)ds\right)d\chi}{\int_{\delta}^{\beta}\omega(s)ds}\right)\int_{\delta}^{\beta}\omega(\chi)d\chi \le \int_{\delta}^{\beta}\xi(\chi)\omega(\chi)d\chi \le \frac{\xi(\beta)}{\beta-\delta}\int_{\delta}^{\beta}(\chi-\delta)\omega(\chi)d\chi + \frac{\xi(\delta)}{\beta-\delta}\int_{\delta}^{\beta}(\beta-\chi)\omega(\chi)d\chi. \tag{9}$$

*Proof.*If ξ is convex, $\forall u \in (\gamma,\beta)$, choose $m \in [\xi'_-(u), \xi'_+(u)]$ and $h(\chi)$ is a line of support for ξ at u, we get

$$h(\chi) = \xi(u) + m(\chi - u) \leq \xi(\chi). \tag{10}$$

Since $\omega(\chi)$ is to be nonnegative function, if the inequality (10) is multiplied by the function $\omega(\chi)$ and is taken integration to $[\delta,\beta]$, then we get

$$\int_\delta^\beta h(\chi)\omega(\chi)d\chi = \xi(u)\int_\delta^\beta \omega(\chi)d\chi + \xi'(u)\int_\delta^\beta (\chi-u)\omega(\chi)d\chi \leq \int_\delta^\beta \xi(\chi)\omega(\chi)d\chi.$$

On the last inequality, the integral is calculated as follow:

$$\int_\delta^\beta (\chi-u)\omega(\chi)d\chi = (\chi-u)\int_\delta^\chi \omega(u)du\Bigg|_\delta^\beta - \int_\delta^\beta \left(\int_\delta^\chi \omega(s)ds\right)d\chi = (\beta-u)\int_\delta^\beta \omega(s)ds - \int_\delta^\beta \left(\int_\delta^\chi \omega(s)ds\right)d\chi. \tag{11}$$

If it is taken u in (11) as follow

$$u = \beta - \frac{\int_\delta^\beta \left(\int_\delta^\chi \omega(s)ds\right)d\chi}{\int_\delta^\beta \omega(s)ds},$$

we get

$$\int_\delta^\beta (\chi-u)\omega(\chi)d\chi = 0.$$

From (10) inequality, we get

$$\xi\left(\beta - \frac{\int_\delta^\beta \left(\int_\delta^\chi \omega(s)ds\right)d\chi}{\int_\delta^\beta \omega(s)ds}\right)\int_\delta^\beta \omega(\chi)d\chi \leq \int_\delta^\beta \xi(\chi)\omega(\chi)d\chi. \tag{12}$$

If you pay attention,

$$0 \leq \frac{\int_\delta^\beta \left(\int_\delta^\chi \omega(s)ds\right)d\chi}{\int_\delta^\beta \omega(s)ds} \leq \int_\delta^\beta 1d\chi = b-a, \quad \delta \leq \beta - \frac{\int_\delta^\beta \left(\int_\delta^\chi \omega(s)ds\right)d\chi}{\int_\delta^\beta \omega(s)ds} \leq \beta$$

Also,

$$\begin{aligned}
\int_\delta^\beta (\chi-u)\omega(\chi)d\chi &= (\chi-u)\int_\beta^\chi \omega(s)ds\Bigg|_\delta^\beta - \int_\delta^\beta \left(\int_\beta^\chi \omega(s)ds\right)d\chi \\
&= -(\delta-u)\int_\beta^\delta \omega(s)ds + \int_\delta^\beta \left(\int_\chi^\beta \omega(s)ds\right)d\chi \\
&= (\delta-u)\int_\delta^\beta \omega(s)ds + \int_\delta^\beta \left(\int_\chi^\beta \omega(s)ds\right)d\chi.
\end{aligned} \tag{13}$$

It is taken

$$u=\delta+\frac{\int\limits_{\delta}^{\beta}\left(\int\limits_{\chi}^{\beta}\omega(s)ds\right)d\chi}{\int\limits_{\delta}^{\beta}\omega(s)ds}$$

in (13), it is obtained

$$\int\limits_{\delta}^{\beta}(\chi-u)\omega(\chi)d\chi=0$$

and from inequality (10), we get

$$\xi\left(\delta+\frac{\int\limits_{\delta}^{\beta}\left(\int\limits_{\chi}^{\beta}\omega(s)ds\right)d\chi}{\int\limits_{\delta}^{\beta}\omega(s)ds}\right)\int\limits_{\delta}^{\beta}\omega(\chi)d\chi\leq\int\limits_{\delta}^{\beta}\xi(\chi)\omega(\chi)d\chi \tag{14}$$

If you pay attention

$$0\leq\frac{\int\limits_{\delta}^{\beta}\left(\int\limits_{\chi}^{\beta}(s)ds\right)d\chi}{\int\limits_{\delta}^{\beta}\omega(s)ds}\leq\beta-\delta$$

$$\delta\leq\delta+\frac{\int\limits_{\delta}^{\beta}\left(\int\limits_{\chi}^{\beta}\omega(s)ds\right)d\chi}{\int\limits_{\delta}^{\beta}\omega(s)ds}\leq\beta.$$

On the other hand, If ξ is convex function for $u\in(\delta,\beta)$ and $\forall\chi\in[\delta,\beta]$ as follow:

$$\xi(\chi)\leq\xi(\beta)\frac{\chi-\delta}{\beta-\delta}+\xi(\delta)\frac{\beta-\chi}{\beta-\delta}. \tag{15}$$

Since $\omega(\chi)$ is to be nonnegative function, if the inequality (15) is multiplied by $\omega(\chi)$ function and is taken integration to $[\delta,\beta]$, then we get

$$\int\limits_{\delta}^{\beta}\xi(x)\omega(\chi)d\chi\leq\frac{\xi(\beta)}{\beta-\delta}\int\limits_{\delta}^{\beta}(\chi-\delta)\omega(\chi)d\chi+\frac{\xi(\delta)}{\beta-\delta}\int\limits_{\delta}^{\beta}(\beta-\chi)\omega(\chi)d\chi \tag{16}$$

If we keep in view of (12), (14) and (16), we get (8) and (9) inequalities. In the other words,

$$\frac{1}{2}\left(\xi\left(\beta-\frac{\int\limits_{\delta}^{\beta}\left(\int\limits_{\delta}^{\chi}\omega(s)ds\right)d\chi}{\int\limits_{\delta}^{\beta}\omega(s)ds}\right)\int\limits_{\delta}^{\beta}\omega(\chi)d\chi+\xi\left(\delta+\frac{\int\limits_{\delta}^{\beta}\left(\int\limits_{\chi}^{\beta}\omega(s)ds\right)d\chi}{\int\limits_{\delta}^{\beta}\omega(s)ds}\right)\int\limits_{\delta}^{\beta}\omega(\chi)d\chi\right)\int\limits_{\delta}^{\beta}\omega(\chi)d\chi \tag{17}$$

$$\leq\int\limits_{\delta}^{\beta}\xi(x)\omega(\chi)d\chi\leq\frac{\xi(\beta)}{\beta-\delta}\int\limits_{\delta}^{\beta}(\chi-\delta)\omega(\chi)d\chi+\frac{\xi(\delta)}{\beta-\delta}\int\limits_{\delta}^{\beta}(\beta-\chi)\omega(\chi)d\chi$$

or

$$\max\left\{\xi\left(\beta-\frac{\int\limits_\delta^\beta\left(\int\limits_\delta^\chi \omega(s)ds\right)d\chi}{\int\limits_\delta^\beta \omega(s)ds}\right),\xi\left(\delta+\frac{\int\limits_\delta^\beta\left(\int\limits_\chi^\beta \omega(s)ds\right)d\chi}{\int\limits_\delta^\beta \omega(s)ds}\right)\right\}\int\limits_\delta^\beta \omega(\chi)d\chi \tag{18}$$

$$\leq\int\limits_\delta^\beta \xi(\chi)\omega(\chi)d\chi\leq\frac{\xi(\beta)}{\beta-\delta}\int\limits_\delta^\beta(\chi-\delta)\omega(\chi)d\chi+\frac{\xi(\delta)}{\beta-\delta}\int\limits_\delta^\beta(\beta-\chi)\omega(\chi)d\chi.$$

Corollary 1. *Let $\xi:[\delta,\beta]\to\mathbb{R}$ be convex function with $\delta<\beta$. If $\omega:[\delta,\beta]\to\mathbb{R}$ is nonnegative, integrable, then for $\lambda\in[0,1]$ the following inequalities hold*

$$\xi\left(\lambda\left(\beta-\frac{\int\limits_\delta^\beta\left(\int\limits_\delta^\chi \omega(s)ds\right)d\chi}{\int\limits_\delta^\beta \omega(s)ds}\right)+(1-\lambda)\left(\delta+\frac{\int\limits_\delta^\beta\left(\int\limits_\chi^\beta \omega(s)ds\right)d\chi}{\int\limits_\delta^\beta \omega(s)ds}\right)\right)$$

$$\leq\int\limits_\delta^\beta \xi(x)\omega(\chi)d\chi\leq\frac{\xi(\beta)}{\beta-\delta}\int\limits_\delta^\beta(x-\delta)\omega(\chi)d\chi+\frac{\xi(\delta)}{\beta-\delta}\int\limits_\delta^\beta(\beta-\chi)\omega(\chi)d\chi \tag{19}$$

Proof. By multiplying the (8) inequality with λ and the (9) with $(1-\lambda)$, and by using with ξ convex function, the proof ends.

Proposition 1. *Let $\omega(\chi)$ function is the symetric with regard to $\chi=\frac{\delta+\beta}{2}$ in under the conditions of the Theorem 8. It is obtained*

$$f\left(\beta-\frac{\int\limits_\delta^\beta\left(\int\limits_\delta^\chi \omega(s)ds\right)d\chi}{\int\limits_\delta^\beta \omega(s)ds}\right)\int\limits_\delta^\beta \omega(\chi)d\chi\leq\int\limits_\delta^\beta \xi(\chi)\omega(\chi)d\chi\leq\frac{\xi(\delta)+\xi(\beta)}{2}\int\limits_\delta^\beta \omega(\chi)d\chi, \tag{20}$$

$$\xi\left(\delta+\frac{\int\limits_\delta^\beta\left(\int\limits_\chi^\beta \omega(s)ds\right)d\chi}{\int\limits_\delta^\beta \omega(s)ds}\right)\int\limits_\delta^\beta \omega(\chi)d\chi\leq\int\limits_\delta^\beta \xi(\chi)\omega(\chi)d\chi\leq\frac{\xi(\delta)+\xi(\beta)}{2}\int\limits_\delta^\beta \omega(\chi)d\chi. \tag{21}$$

Proof. Firstly we going to apply $\omega(\chi)=\omega(\delta+\beta-\chi)$ (such as $\omega(x)$ is the symetric respect to $\frac{\delta+\beta}{2}$) on the left side of (8) inequality. For this, it is obtained

$$\beta-\frac{\int\limits_\delta^\beta\left(\int\limits_\delta^\chi \omega(s)ds\right)d\chi}{\int\limits_\delta^\beta \omega(s)ds}-\delta-\frac{\int\limits_\delta^\beta\left(\int\limits_\chi^\beta \omega(s)ds\right)d\chi}{\int\limits_\delta^\beta \omega(s)ds}=\beta-\delta-\left(\frac{\int\limits_\delta^\beta\left(\int\limits_\delta^\chi \omega(s)ds\right)d\chi+\int\limits_\delta^\beta\left(\int\limits_\chi^\beta \omega(s)ds\right)d\chi}{\int\limits_\delta^\beta \omega(s)ds}\right)$$

$$=\beta-\delta-\frac{\int\limits_\delta^\beta\left(\int\limits_\delta^\beta \omega(s)ds\right)d\chi}{\int\limits_\delta^\beta \omega(s)ds}=\beta-\delta-\frac{(\beta-\delta)\int\limits_\delta^\beta \omega(s)ds}{\int\limits_\delta^\beta \omega(s)ds}=0.$$

This shows that

$$\xi\left(\beta-\frac{\int_{\delta}^{\beta}\left(\int_{\delta}^{\chi}\omega(s)ds\right)d\chi}{\int_{\delta}^{\beta}\omega(s)ds}\right)=\xi\left(\delta+\frac{\int_{\delta}^{\beta}\left(\int_{\chi}^{\beta}\omega(s)ds\right)d\chi}{\int_{\delta}^{\beta}\omega(s)ds}\right). \tag{22}$$

On the other hand, if we use the changed value with $\chi=\delta+\beta-u$ on the right side of (8) inequality, we get

$$\frac{\xi(\beta)}{\beta-\delta}\int_{\delta}^{\beta}(\chi-\delta)\omega(\chi)d\chi+\frac{\xi(\delta)}{\beta-\delta}\int_{\delta}^{\beta}(\beta-\chi)\omega(\chi)d\chi=\frac{\xi(\beta)}{\beta-\delta}\int_{\delta}^{\beta}(\beta-\chi)\omega(\chi)d\chi+\frac{\xi(\delta)}{\beta-\delta}\int_{\delta}^{\beta}(\chi-\delta)\omega(\chi)d\chi. \tag{23}$$

Thus, (8) inequality is as follows by using (22) and (23) equalities:

$$\xi\left(\delta+\frac{\int_{\delta}^{\beta}\left(\int_{\chi}^{\beta}\omega(s)ds\right)d\chi}{\int_{\delta}^{\beta}\omega(s)ds}\right)\leq\int_{\delta}^{\beta}\xi(\chi)\omega(\chi)d\chi\leq\frac{\xi(\beta)}{\beta-\delta}\int_{\delta}^{\beta}(\beta-\chi)\omega(\chi)d\chi+\frac{\xi(\delta)}{\beta-\delta}\int_{\delta}^{\beta}(\chi-\delta)\omega(\chi)d\chi. \tag{24}$$

If we gather (24) and (9) and then multiply with $\frac{1}{2}$, we get (21) inequality. Similarly, (20) inequality can be obtained inequality.

Corollary 2. *1.If we take $\omega(\chi)=1$ to (8) and (9), we get*

$$\xi\left(\frac{\delta+\beta}{2}\right)\leq\frac{1}{\beta-\delta}\int_{\delta}^{\beta}\xi(\chi)d\chi\leq\frac{\xi(\delta)+\xi(\beta)}{2}$$

2.If we take $\omega(\chi)=(\beta-\chi)^{\alpha-1}$ and $\omega(\chi)=(\chi-\delta)^{\alpha-1}$ to (8) respectly ,we get (see [6, Theorem 2.1])

$$\xi\left(\frac{\alpha\delta+\beta}{\alpha+1}\right)\leq\frac{\Gamma(\alpha+1)}{(\beta-\delta)^{\alpha}}J^{\alpha}_{\delta+}\xi(\beta)\leq\frac{\alpha\xi(\delta)+\xi(\beta)}{\alpha+1}, \tag{25}$$

$$\xi\left(\frac{\delta+\alpha\beta}{\alpha+1}\right)\leq\frac{\Gamma(\alpha+1)}{(\beta-\delta)^{\alpha}}J^{\alpha}_{\beta-}\xi(\delta)\leq\frac{\xi(\delta)+\alpha\xi(\beta)}{\alpha+1}, \tag{26}$$

3.If we take $\omega(\chi)=(\beta-\chi)^{\alpha-1}$ or $\omega(\chi)=(\chi-\delta)^{\alpha-1}$ to (9) respectly, ,we get (see [7, Theorem 2.1])

$$\xi\left(\frac{\delta+\alpha\beta}{\alpha+1}\right)\leq\frac{\Gamma(\alpha+1)}{(\beta-\delta)^{\alpha}}J^{\alpha}_{\delta+}\xi(\beta)\leq\frac{\alpha\xi(\delta)+\xi(\beta)}{\alpha+1}, \tag{27}$$

$$\xi\left(\frac{\alpha\delta+\beta}{\alpha+1}\right)\leq\frac{\Gamma(\alpha+1)}{(\beta-\delta)^{\alpha}}J^{\alpha}_{\beta-}\xi(\delta)\leq\frac{\xi(\delta)+\alpha\xi(\beta)}{\alpha+1}, \tag{28}$$

4.If we take $\omega(\chi)=(\chi-\delta)^{\alpha-1}+(\beta-\chi)^{\alpha-1}$ to (8) and (9), we get

$$\xi\left(\frac{\delta+\beta}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(\beta-\delta)^{\alpha}}\left[J^{\alpha}_{\beta-}\xi(\delta)+J^{\alpha}_{\delta+}\xi(\beta)\right]\leq\frac{\xi(\delta)+\xi(\beta)}{2}$$

5.If we take $\omega(\chi)=(\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon-1}+(\chi-\delta)^{\varepsilon}(\beta-\chi)^{\alpha-\varepsilon-1}$ to (8) and (9), we get [11, Theorem 2.1]

*Proof.*It is taken $g(\chi)=(\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon-1}+(\chi-\delta)^{\varepsilon}(\beta-\chi)^{\alpha-\varepsilon-1}$ with $\alpha=\varepsilon+1$ for $\varepsilon=0,1,2\cdots$ on the left of inequality (8) and then by change of the order of integration, we have

$$
\begin{aligned}
&\xi\left(\beta-\frac{\int\limits_{\delta}^{\beta}\left(\int\limits_{\delta}^{\chi}\begin{pmatrix}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}\\+(s-\delta)^{\varepsilon}(\beta-s)^{\alpha-\varepsilon-1}\end{pmatrix}ds\right)d\chi}{\left(\int\limits_{\delta}^{\beta}\begin{pmatrix}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}\\+(s-\delta)^{\varepsilon}(\beta-s)^{\alpha-\varepsilon-1}\end{pmatrix}ds\right)}\right)\left(\int\limits_{\delta}^{\beta}\begin{pmatrix}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}\\+(s-\delta)^{\varepsilon}(\beta-s)^{\alpha-\varepsilon-1}\end{pmatrix}ds\right)\\
&=\xi\left(\beta-\frac{\int\limits_{\delta}^{\beta}\left(\int\limits_{s}^{\beta}\begin{pmatrix}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}\\+(s-\delta)^{\varepsilon}(\beta-s)^{\alpha-\varepsilon-1}\end{pmatrix}d\chi\right)ds}{\left(\int\limits_{\delta}^{\beta}\begin{pmatrix}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}\\+(s-\delta)^{\varepsilon}(\beta-s)^{\alpha-\varepsilon-1}\end{pmatrix}ds\right)}\right)\left(\int\limits_{\delta}^{\beta}\begin{pmatrix}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}\\+(s-\delta)^{\varepsilon}(\beta-s)^{\alpha-\varepsilon-1}\end{pmatrix}ds\right)\\
&=\xi\left(\beta-\frac{\int\limits_{\delta}^{\beta}(\beta-s)^{\varepsilon+1}(s-\delta)^{\alpha-\varepsilon-1}ds+\int\limits_{\delta}^{\beta}(s-\delta)^{\varepsilon}(\beta-s)^{\alpha-\varepsilon}ds}{\int\limits_{\delta}^{\beta}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}ds+\int\limits_{\delta}^{\beta}(s-\delta)^{\varepsilon}(\beta-s)^{\alpha-\varepsilon-1}ds}\right)\left(\int\limits_{\delta}^{\beta}\begin{pmatrix}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}\\+(s-\delta)^{\varepsilon}(\beta-s)^{\alpha-\varepsilon-1}\end{pmatrix}ds\right)\\
&=\xi\left(\beta-\frac{(\beta-\delta)\left[\int\limits_{0}^{1}\iota^{\varepsilon+1}(1-\iota)^{\alpha-\varepsilon-1}d\iota+\int\limits_{0}^{1}\iota^{\varepsilon}(1-\iota)^{\alpha-\varepsilon}d\iota\right]}{2\int\limits_{0}^{1}\iota^{\varepsilon}(1-\iota)^{\alpha-\varepsilon-1}d\iota}\right)2(\beta-\delta)^{\alpha}\int\limits_{0}^{1}\iota^{\varepsilon}(1-\iota)^{\alpha-\varepsilon-1}d\iota.
\end{aligned}
\tag{29}
$$

If we continue on the other side of (8) inequality,

$$
\begin{aligned}
&\int\limits_{\delta}^{\beta}(\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon-1}\xi(\chi)d\chi+\int\limits_{\delta}^{\beta}(\chi-\delta)^{\varepsilon}(\beta-\chi)^{\alpha-\varepsilon-1}\xi(\chi)d\chi\\
&\leq\frac{\xi(\beta)}{\beta-\delta}\left[\int\limits_{\delta}^{\beta}(\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon}d\chi+\int\limits_{\delta}^{\beta}(\chi-\delta)^{\varepsilon+1}(\beta-\chi)^{\alpha-\varepsilon-1}d\chi\right]\\
&\quad+\frac{\xi(\delta)}{\beta-\delta}\left[\int\limits_{\delta}^{\beta}(\beta-\chi)^{\varepsilon+1}(\chi-\delta)^{\alpha-\varepsilon-1}d\chi+\int\limits_{\delta}^{\beta}(\chi-\delta)^{\varepsilon}(\beta-\chi)^{\alpha-\varepsilon}d\chi\right]\\
&=(\beta-\delta)^{\alpha}\left[\xi(\delta)+\xi(\beta)\right]\int\limits_{0}^{1}\iota^{\varepsilon}(1-\iota)^{\alpha-\varepsilon-1}d\iota.
\end{aligned}
\tag{30}
$$

By using Beta function and combine with (29) and (30), the proof is completed.

6.If we take $\omega(\chi)=(\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon-1}$ *to (8), we have [13, Theorem 2.1]*

Proof. It is taken $\omega(\chi)=(\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon-1}$ with $\alpha=\varepsilon+1$ for $\varepsilon=0,1,2\cdots$ on the left of inequality (8) and then by change of the order of integration, we have

$$\xi\left(\beta-\frac{\int_{\delta}^{\beta}\left(\int_{\delta}^{\chi}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}ds\right)d\chi}{\int_{\delta}^{\beta}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}ds}\right)\left(\int_{\delta}^{\beta}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}ds\right)$$

$$=\xi\left(\beta-\frac{\int_{\delta}^{\beta}\left(\int_{s}^{\beta}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}d\chi\right)ds}{\int_{\delta}^{\beta}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}ds}\right)\left(\int_{\delta}^{\beta}(\beta-s)^{\varepsilon}(s-\delta)^{\alpha-\varepsilon-1}ds\right)$$

$$=\xi\left(\beta-\frac{(\beta-\delta)\int_{0}^{1}\iota^{\varepsilon+1}(1-\iota)^{\alpha-\varepsilon-1}d\iota}{\int_{0}^{1}\iota^{\varepsilon}(1-\iota)^{\alpha-\varepsilon-1}d\iota}\right)\left(\int_{0}^{1}\iota^{\varepsilon}(1-\iota)^{\alpha-\varepsilon-1}d\iota\right)$$

$$=\xi\left(\frac{(\beta-\delta)B(\varepsilon+2,\alpha-\varepsilon)}{B(\varepsilon+1,\alpha-\varepsilon)}\right)B(\varepsilon+1,\alpha-\varepsilon)$$

$$=\xi\left(\frac{(\alpha-\varepsilon)\beta+(\varepsilon+1)\delta}{\alpha+1}\right)B(\varepsilon+1,\alpha-\varepsilon). \tag{31}$$

If we continue on the other side of (8) inequality,

$$\int_{\delta}^{\beta}(\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon-1}\xi(\chi)d\chi$$

$$\leq\frac{\xi(\beta)}{\beta-\delta}\int_{\delta}^{\beta}(\beta-\delta)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon}d\chi+\frac{\xi(\delta)}{\beta-\delta}\int_{\delta}^{\beta}(\beta-\delta)^{\varepsilon+1}(\chi-\delta)^{\alpha-\varepsilon-1}d\chi$$

$$=(\beta-\delta)^{\alpha}\left[\xi(\beta)\int_{0}^{1}\iota^{\varepsilon}(1-\iota)^{\alpha-\varepsilon}d\iota+\xi(\delta)\int_{0}^{1}\iota^{\varepsilon+1}(1-\iota)^{\alpha-\varepsilon-1}d\iota\right]$$

$$=(\beta-\delta)^{\alpha}\left[\xi(\beta)B(\varepsilon+1,\alpha-\varepsilon+1)+\xi(\delta)B(\varepsilon+2,\alpha-\varepsilon)\right]. \tag{32}$$

After we assemble the inequalities of (31) and (32), the proof is completed.

7. *If we take $\omega(\chi)=(\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon-1}$ to (9), we obtain [14, Theorem 2.1]*

Proof. When it is writen by $\omega(\chi)=(\beta-\chi)^{\varepsilon}(\chi-\delta)^{\alpha-\varepsilon-1}$, $\alpha\in(\varepsilon,\varepsilon+1]$, $\varepsilon=0,1,2,...,$ on (9) and continued with the same way in the proof of (6) , this proof is completed.

Remark. 1. If we have gathered two side of (25)-(26) or (27)-(28) inequalities and then have multiplied with $\frac{1}{2}$, we get [8, Theorem 6].
2. If we have gathered two side of the inequalities of (6)-(7) in Corollary 2 and then have multiplied with $\frac{1}{2}$, we get [15, Theorem 6].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] T. Abdeljawad, On conformable fractional calculus, Journal of Computational and Applied Mathematics, 279 (2015), 57-66, https://doi.org/10.1016/j.cam.2014.10.016.
[2] L. Fejér, Über die Fourierreihen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 24 (1906), pp.369-390.
[3] Hadamard, J., Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann, J. Math. Pures Appl., 58 (1893), 171-215.
[4] Hermite, Ch., Sur deux limites d'une intégrale définie, Mathesis, 3 (1883), 82–83.
[5] İ. İşcan, Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals, Stud. Univ. Babeş-Bolyai Math. 60(3) (2015), 355-366.
[6] M. Kunt ,D. Karapınar ,S. Turhan ,İ. İşcan, The left Rieaman-Liouville fractional Hermite-Hadamard type inequalities for convex functions, RGMIA Research Report Collection, 20, Article 101 (2017), 8 pp.
[7] M.Kunt ,D. Karapinar ,S. Turhan ,İ. İşcan , "The right Riemann-Liouville fractional Hermite-Hadamard type inequalities for convex functions", Journal of Inequalities and Special Functions, vol.9 (2018), pp.45-57.
[8] M. Kunt, İ. İşcan, S. Turhan, D. Karapınar, Improvement of fractional Hermite-Hadamard type inequality and some new fractional midpoint type inequalities for convex functions, Miskolc Mathematical Notes 19.2 (2018), 1007-1017.
[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations. Elsevier, Amsterdam (2006).
[10] M. Sarıkaya ,E. Set , Başak N. Yaldız, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling 57 (2013) 2403-2407.
[11] E. Set ,M. Z. Sarikaya and A. Gözpınar, Some Hermite-Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities. Creat. Math. Inform 26.2 (2017), 221-229.
[12] E. Set, İ. Mumcu, Hermite-Hadamard-Fejér type inequalities for conformable fractional integrals, Available online from: https://www.researchgate.net/publication/303382008.
[13] S. Turhan, The left conformable fractional Hermite-Hadamard type inequalities for convex functions, Filomat,33:8 (2019), 2417-2430.
[14] S. Turhan, İ. İşcan, M. Kunt, The right conformable fractional Hermite-Hadamard type inequalities for convex functions, Available online from: https://www.researchgate.net/publication/
[15] S. Turhan, İ. İşcan, M. Kunt, Improvement of Conformable Fractional Hermite-Hadamard Type Inequality For Convex Functions and Some New Conformable Fractional Midpoint Type Inequalities, Malaya Journal of Matematik, 8 (3) (2020), 753-760.
[16] Y. Zhou, Basic theory of fractional differential equations, World Scientific, New Jersey (2014).