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On some special ruled surfaces

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Abstract: In this paper, regarding α and β, we examine some special ruled surfaces in (α,β)-type normal almost contact metric (briefly a.c.m.) manifolds whose base curves are almost contact curves which are not geodesic. By selecting the base curve and the ruling of these ruled surfaces, we obtain important theorems and corollaries.

Keywords: Normal almost contact manifold; Curves; Ruled surfaces.

1 Introduction

The ruled surfaces are a beneficial subject of differential geometry. A ruled surface is formed by a 1-parameter family of straight lines. Ruled surfaces vary in selected curves, specified spaces and the use of different frames in these spaces. Looking at this framework, we see its applications in many areas of science. So, many researchers, have investigated ruled surfaces, for example, geometers, physicists and engineers. The ruled surface geometry is fundamental in various fields of computer-aided design (CAD).

Within the literature, there are many studies on ruled surfaces published by geometers. Some of them are the articles [1, 2,5,7,11,12,15,16,17,18,19].

We have two main references. First, Karacan et. al investigated a ruled surface in Sasakian manifolds, [10]. Second, Izumiya and Takeuchi investigated Bertrand curves and cylindrical helices on ruled surfaces, [9].

In this paper, we give basic notations, definetions and theorems that will provide convenience in our study and then we identify some special ruled surfaces that are generated by some special curves in $3-$dimensional normal a.c.m. manifold. Finally, we obtain some theorems and corollaries regarding these ruled surfaces.

2 Preliminaries

Let

–a $3-$dimensional manifold M be a Riemannian manifold given with a metric g
–the dual basis of an orthogonal $\{e_1,e_2,e_3\}$ basis on M be $\left\{\theta^1,\theta^2,\theta^3\right\}$ where $\theta^i(e_j)=\delta_{ij}$, $(1\leq i,j\leq 3)$,
–the Levi-Civita connection on M be ∇,
–η, φ and ξ be defined $\eta:\mathfrak{X}(M)\longrightarrow C^\infty(M,\mathbf{R})$, $\varphi:\mathfrak{X}(M)\longrightarrow\mathfrak{X}(M)$ and $\xi\in\mathfrak{X}(M)$, respectively,
–$\zeta:I\longrightarrow M$, $\zeta(s)=(\zeta_1(s),\zeta_2(s),\zeta_3(s))$ be a regular unite speed curve given by arc length s.

Accordingly, any two vectors $x=\sum\limits_{i=1}^{3}\theta^i(x)e_i$, $y=\sum\limits_{i=1}^{3}\theta^i(y)e_i$ of $\mathfrak{X}(M)$, we can give the following concepts.

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Definition 1. *The inner product and the vector product are defined by*

$$g(x,y)=\sum_{i=1}^{3}\theta^{i}(x)\theta^{i}(y),$$

and

$$x\wedge y=\sum_{(i\neq i)=1}^{3}\theta^{i}(x)\theta^{j}(y)e_i\wedge e_j, \tag{1}$$

respectively. According to (1)

$$e_i\wedge e_j=s(\sigma)e_k,$$

where $s(\sigma)$ is the sign of permutations $\sigma\in\mathbf{S}_3$.

Definition 2. [13] *Let a curve ζ in M be a regular curve $\zeta'=\dfrac{\mathrm{d}\zeta}{\mathrm{d}s}$ is non-zero. If ζ' and ζ'' are linearly independent at each point throughout itself, in that case, ζ is named a Frenet curve .*

Definition 3. *Given a unit speed Frenet curve ζ. Then $T=\zeta'=\dfrac{d\zeta}{ds}$ is named the unit tangent vector field of ζ , $N=\dfrac{T'}{\|T'\|}$ is named the normal vector field of ζ and $B=T\wedge N$ is named the binormal vector field of ζ. Briefly, we say that $\{T,N,B\}$ is the Frenet-Serret triplet on ζ.*

Theorem 1. [20] *Given a unit speed Frenet curve ζ and let its Frenet -Serret vector fields are $\{T,N,B\}$. The Frenet formulas of ζ are*

$$T'=\kappa N,\quad N'=\kappa T+\tau B,\quad B'=\tau N$$

where $\kappa=\|T'\|$ and $\tau=g(N',B)$ are geodesic curvature and geodesic torsion curvature of ζ, respectively.

Definition 4. *The terms $\{\kappa,\tau,T,N,B\}$ obtained by means of a curve ζ are named the Frenet components.*

Definition 5. [8] *Given three tensors (φ,ξ,η) such that*

$$\eta(\varphi)=0,\quad \varphi^2x=-x+\eta(x)\xi,\quad \eta(\xi)=1.$$

Then (φ,ξ,η) is named an almost contact structure on M, and M is also named an almost contact manifold.

Definition 6. [8] *If the expression $\eta\wedge(d\eta)\neq 0$ exists, then we name η a contact form on M and $D_m=\{x\in T_M(m):\eta(x)=0,\ m\in M\}$ is named contact distribution.*

Definition 7. [8], [14] *If a metric g provides the following expression*

$$g(x,y)=g(\varphi x,\varphi y)+\eta(x)\eta(y)$$

then (φ,ξ,η,g) is a.c.m structure and with that, the quintup $(M;\varphi,\xi,\eta,g)$ is named an a.c.m. manifold. We have

$$\eta(x)=g(x,\xi),\quad \Phi(x,y)=g(x,\varphi y)$$

where Φ is a $2-$form on M.

Definition 8. [14] *A transform J defined as*

$$\begin{aligned} J:\in\mathfrak{X}(M)\times\mathbf{R} &\longrightarrow \in\mathfrak{X}(M)\times\mathbf{R}\\ \left(x,\rho\frac{\mathrm{d}}{\mathrm{d}s}\right) &\longrightarrow J\left(x,\rho\frac{\mathrm{d}}{\mathrm{d}s}\right)=\left(\varphi x-\rho\xi,\eta(x)\frac{\mathrm{d}}{\mathrm{d}s}\right)\end{aligned}$$

is named almost contact structure, where s and ρ are a coordination function of $\mathbf{R}$ and a differentiable function on $M\times\mathbf{R}$, respevtively.

Definition 9.[4] *A torsion* $[\varphi,\varphi]$ *of* φ *defined as*

$$[\varphi,\varphi](x,y)=[\varphi x,\varphi y]+\varphi^2[x,y]-\varphi[\varphi x,y]-\varphi[x,\varphi y]$$

is named Nijenhuis torsion if the almost contact structure J is integrable, namely

$$2\mathrm{d}\eta(x,y)\xi=-[\varphi,\varphi](x,y),$$

then the triplet $(M;\varphi,\xi,\eta)$ *is named normal or simply M is named normal almost contact manifold.*

Proposition 1.[14] *If M is an a.c.m. manifold then, we have*

$$g(\varphi\nabla_x\xi,y)\xi-\eta(y)\varphi\nabla_x\xi=(\nabla_x\varphi)y. \tag{2}$$

Theorem 2.[14] *An a.c.m. manifold M is normal iff*

$$\varphi\nabla_x\xi=\nabla_{\varphi x}\xi. \tag{3}$$

As a result of (3)*, we can write the following equations*

$$-\beta\varphi^2(x)-\alpha\varphi(x)=\beta(x-\eta(x)\xi)-\alpha\varphi(x)=\nabla_x\xi, \tag{4}$$

here $\beta=\frac{1}{2}Trace(\nabla\xi)$, $\quad\alpha=\frac{1}{2}Trace(\varphi\nabla\xi)$.

As we will indicate below, we will divide a $3-$dimensional normal a.c.m. manifold M into classes with special conditions. The pair (α,β) is an indicator showing the classes of M.

*Remark.*Using (3) or (4), we obtain $\nabla_\xi\xi=0$. If $\dfrac{\mathrm{d}\zeta}{\mathrm{d}s}=\xi(\zeta(s))$, then ζ is a integral curve that is geodesic.

Using (2) and (4), we have

$$(\nabla_x\varphi)y=\alpha(g(x,y)\xi-\eta(y)x)+\beta(g(\varphi x,y)\xi-\eta(y)\varphi x), \tag{5}$$

and there are always

$$2\beta\alpha+\xi[\alpha]=0 \tag{6}$$

on a normal a.c.m. manifold M, [8]. We consider (6), $\beta=0$ when α is a constant not zero. Particularly, we can classify a normal a.c.m. manifold M as follows:

– M is a cosymplectic (or a co-Kähler) manifold if $\beta=\alpha=0$.
– M is a quasi-Sasakian manifold if $\beta=0$ and $\xi[\alpha]=0$.
– M is a $\alpha-$ Sasakian manifold if $\beta=0$ and α is a non-zero constant.
– M is an $\beta-$ Kenmotsu manifold if β is a non-zero constant and $\alpha=0$.

Proposition 2.*If M is a cosymplectic (or a co-Kähler) manifold, in that case,we have*

$$\nabla_x\eta(y)=\eta(\nabla_x y),\quad\nabla_x\varphi y=\varphi\nabla_x y.$$

Proposition 3.*If M is a quasi-Sasakian manifold and* α *is not a constant, in that case we have*

$$\xi[\alpha]=g(\xi,\ grad\alpha)=0.$$

According to the statements, an α-Sasakian manifold and a β- Kenmotsu manifold are the type $(\alpha,0)$ and the type $(0,\beta)$ of the $3-$dimensional normal a.c.m. manifold M, respectively. We say that an 1-Sasakian manifold is Sasakian manifold and an 1-Kenmotsu manifold is Kanmotsu manifold.

Proposition 4. *Given an orthogonal φ-basis $\{x, y=\varphi x, \xi\}$ on a normal a.c.m. manifold M, then*

$$\beta = -\eta\left(\nabla_x x\right).$$

Proof. Using (5), we obtain $-\beta\xi = \left(\nabla_x \varphi\right) y = \eta\left(\nabla_x x\right)\xi$.

Corollary 1. *Let M be a normal a.c.m. manifold, let $x \in D_m$ be a unit normal vector field and let $\beta = 0$, then $\nabla_x x$ and ξ are orthogonal.*

Corollary 2. *Let M be a normal a.c.m. manifold. For $\forall x \in \mathfrak{X}(M)$,*

$$\nabla_\xi \varphi x = \varphi \nabla_\xi x, \tag{7}$$

and

$$[\varphi x, \xi] = \varphi\,[x, \xi]. \tag{8}$$

Corollary 3. [14] *Given a normal a.c.m. manifold M, then we have*

$$\mathrm{d}\eta = \alpha\Phi.$$

Corollary 4. *If M is a cosymplectic(or a $\beta-$ Kenmotsu) manifold, in that case, we have*

$$\mathrm{d}\eta = 0, \quad \eta \wedge (\mathrm{d}\eta) = 0,$$

namely, η is not a contact form on these manifolds.

3 Some Special Curves in a 3-Dimensional Normal a.c.m. Manifold

Definition 10. [3], [8] *Let a curve ζ be a Frenet curve in an a.c.m. manifold M. If $\eta(\zeta')=0$, in that case, ζ is named an almost contact curve. In particular, if η is a contact form, in that case, ζ is named a Legendre curve.*

Let the curve ζ be an almost contact curve by the arc length s on a normal a.c.m. manifold M. Given the $\varphi-$ basis of the curve ζ $\{e_1=\zeta', e_2=\varphi\zeta', e_3=\xi\}$, then their derivative changes are the following equations:

$$\begin{bmatrix} \nabla_{\zeta'}\zeta' \\ \nabla_{\zeta'}\varphi\zeta' \\ \nabla_{\zeta'}\xi \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon & -\beta \\ -\varepsilon & 0 & \alpha \\ \beta & -\alpha & 0 \end{bmatrix} \begin{bmatrix} \zeta' \\ \varphi\zeta' \\ \xi \end{bmatrix}, \tag{9}$$

where $\varepsilon = g\left(\varphi\zeta', \nabla_{\zeta'}\zeta',\right)$, $\beta = \frac{1}{2}\mathrm{Trace}\left(\nabla\xi\right)$, $\alpha = \frac{1}{2}\mathrm{Trace}\left(\varphi\nabla\xi\right)$. The anti-symmetric matrix in (9) gives the connection forms of the velocity vector ζ' whose associated basis is $\{\zeta', \varphi\zeta', \xi\}$. We can make this matrix corresponding to a vector field in the $\in \mathfrak{X}(M)$. Namely, let

$$\omega_{ij}\left(\zeta'\right) = \begin{bmatrix} 0 & \varepsilon & -\beta \\ -\varepsilon & 0 & \alpha \\ \beta & -\alpha & 0 \end{bmatrix} \in A_{3\times 3},$$

so, there is a linear isomorphism as follows;

$$\begin{array}{rcl} \omega: \; A_{3\times 3} & \longrightarrow & \mathfrak{X}(M) \\ \omega_{ij}\left(\zeta'\right) & \longrightarrow & \omega\left(\omega_{ij}\left(\zeta'\right)\right) = \alpha\zeta' + \beta\varphi\zeta' + \varepsilon\xi \end{array} \tag{10}$$

where $A_{3\times 3}$ represents the family of anti-symmetric matrices. We will, in short, show the vector $\alpha\zeta' + \beta\varphi\zeta' + \varepsilon\xi$ as ω. For the curve ζ, $T = \zeta'$ and

$$\kappa N = \nabla_{\zeta'}\zeta' = \varepsilon\varphi\zeta' - \beta\xi. \tag{11}$$

Accordingly, we have

$$\kappa = \sqrt{\varepsilon^2 + \beta^2}.$$

On the other hand, the following equations are obtained;

$$\nabla_{\zeta'}N = \left(\frac{\varepsilon}{\kappa}\right)'\varphi\zeta' + \frac{\varepsilon}{\kappa}\nabla_{\zeta'}\varphi\zeta' - \left(\frac{\beta}{\kappa}\right)'\varphi\zeta' - \frac{\beta}{\kappa}\nabla_{\zeta'}\xi = -\kappa T + \tau B,$$

$$\tau B = \left[\left(\frac{\varepsilon}{\kappa}\right)' - \frac{\beta\alpha}{\kappa}\right]\varphi\zeta' + \left[\left(-\frac{\beta}{\kappa}\right)' - \frac{\varepsilon\alpha}{\kappa}\right]\xi, \tag{12}$$

and

$$\left[\left(\frac{\varepsilon}{\kappa}\right)'\left(-\frac{\beta}{\kappa}\right) - \frac{\varepsilon}{\kappa}\left(-\frac{\beta}{\kappa}\right)'\right]^2 = \left[\left(\frac{\varepsilon}{\kappa}\right)'\right]^2 + \left[\left(-\frac{\beta}{\kappa}\right)'\right]^2. \tag{13}$$

If the norm of both sides of (12) is taken, we can obtain the following equation,

$$\tau = \left|\alpha - \sqrt{\left[\left(\frac{\beta}{\kappa}\right)'\right]^2 + \left[\left(\frac{\varepsilon}{\kappa}\right)'\right]^2}\right|.$$

So, we can obtain the Frenet-Serret vector fields of ζ as follows;

$T = \zeta' = e_1,$

$N = \dfrac{\varepsilon}{\kappa}\varphi\zeta' - \dfrac{\beta}{\kappa}\xi,$

$B = \dfrac{1}{\tau}\left[\left(\dfrac{\varepsilon}{\kappa}\right)' + \dfrac{\beta\alpha}{\kappa}\right]\varphi\zeta' + \dfrac{1}{\tau}\left[\left(-\dfrac{\beta}{\kappa}\right)' - \dfrac{\varepsilon\alpha}{\kappa}\right]\xi.$

With a simple calculation, we can see that

$$\xi = -\frac{\beta}{\kappa}N + \frac{1}{\tau}\left[\left(-\frac{\beta}{\kappa}\right)' - \frac{\varepsilon\alpha}{\kappa}\right]B.$$

4 Some Special Ruled Surface in a 3-Dimensional Normal a.c.m. Manifold

Let a curve ζ be a non-geodesic almost contact curve $(\kappa \neq 0)$ by the arc length parameter s in a normal a.c.m. manifold M.

A ruled surface $\Omega(s,v)$ is shown by

$$\begin{aligned} \Omega : I \times \mathbf{R} &\longrightarrow M \\ (s,v) &\longrightarrow \Omega(s,v) = \zeta(s) + v\Gamma(s) \end{aligned} \tag{14}$$

where, $\zeta(s)$ is named the directrix curve and $\Gamma(s)$ is the unit normal represents a space curve which representing the director vector of a ruling forming the ruled surface Ω. The ratio of the shortest distance between two neighboring main rulings of a ruled surface to the angle between these two neighboring rulings is named the distribution parameter of the ruled surface.

The curve intersecting each ruling perpendicular is named the orthogonal trajectory of the ruled surface. The foot of the common perpendicular line of the two neighboring rulings of the ruled surface on the main ruling is named the center or striction point. Thus, the main ruling of the ruled surface forms the surface along the base curve, while the geometric loci of the striction points form the striction line of the ruled surface, [6].

The distribution parameter of the ruled surface Ω is

$$P_x = \frac{\det(\zeta', \Gamma, \Gamma')}{\|\Gamma'\|^2},$$

and its striction line is

$$\bar{\zeta}(s) = \zeta(s) - \frac{g(\zeta'(s), \Gamma'(s))^2}{\|\Gamma'(s)\|}\Gamma(s), \tag{15}$$

where $\Gamma'(s) = \nabla_{\zeta'}\Gamma$. The basis φ related to ζ' is $\{e_1 = \zeta', e_2 = \varphi\zeta', e_3 = \xi\}$. We can express the vector $\Gamma(s)$ concerning the $\varphi-$basis as follows;

$$\Gamma(s) = \gamma_1\zeta'(s) + \gamma_2\varphi\zeta'(s) + \gamma_3\xi(s) \tag{16}$$

where $g(\Gamma, \Gamma) = 1$. Moreover, the derivative of $\Gamma(s)$ concerning s becomes

$$\Gamma'(s) = \nabla_{\zeta'}\Gamma = \gamma_1\nabla_{\zeta'}\zeta' + \gamma_2\nabla_{\zeta'}\varphi\zeta' + \gamma_3\nabla_{\zeta'}\xi \tag{17}$$

Substituting (9) into (17) give

$$\Gamma'(s) = (\gamma_3\varepsilon - \gamma_2\beta)\zeta' + (\gamma_1\varepsilon - \gamma_3\alpha)\varphi\zeta' + (\gamma_2\alpha - \gamma_1\beta)\xi. \tag{18}$$

Considering (10), we get (18) as follows

$$\Gamma' = \omega \wedge \Gamma. \tag{19}$$

The vertical projection length of the vector ω on the vector Γ becomes

$$\ell = g(\omega, \Gamma) = \|\omega\|\cos\theta = \gamma_1\alpha + \gamma_2\beta + \gamma_3\varepsilon \tag{20}$$

where θ is the angle between ω and Γ. Thus, we can obtain that

$$\Gamma' \wedge \Gamma = \ell\Gamma - \omega. \tag{21}$$

Furthermore, we have

$$\frac{\partial\Omega(s,v)}{\partial s} \wedge \frac{\partial\Omega(s,v)}{\partial v} = \left(\zeta'(s) + v\Gamma'(s)\right) \wedge \Gamma(s) = \zeta'(s) \wedge \Gamma(s) + v\Gamma'(s) \wedge \Gamma(s). \tag{22}$$

A point (s_0, v_0) of $\Omega(s,v)$ is a singular point iff

$$\zeta'(s_0) \wedge \Gamma(s_0) + v_0\Gamma'(s_0) \wedge \Gamma(s_0) = 0.$$

Additionally, the ruled surface $\Omega(s,v)$ is cylindrical iff

$$\Gamma' \wedge \Gamma = 0 \tag{23}$$

[9]. Substituting (21) into (22) gives the following

$$\frac{\partial\Omega(s,v)}{\partial s} \wedge \frac{\partial\Omega(s,v)}{\partial v} = \zeta' \wedge \Gamma + v(\ell\Gamma - \omega). \tag{24}$$

On the other hand, using (19) and (21), we can obtain the distribution parameter P_x as follows:

$$P_x = \frac{\det(\zeta', \Gamma, \Gamma')}{\|\Gamma'\|^2} = \frac{\alpha - \gamma_1 \ell}{\|\Gamma'\|^2}. \tag{25}$$

Theorem 3. *A ruled surface $\Omega(s,v) = \zeta(s) + v\Gamma(s)$ is developable iff*

$$\alpha - \gamma_1 \ell = 0. \tag{26}$$

Proof. If $\Omega(s,v)$ is developable, in that case, $P_x = 0$, [6]. Thus, if $\det(\zeta', \Gamma, \Gamma') = 0$ in (25), then $P_x = 0$. Hence,

$$0 = \det(\zeta', \Gamma, \Gamma') = g(\zeta', \Gamma \wedge \Gamma') = g(\zeta', \omega - \ell\Gamma) = \alpha - \gamma_1 \ell.$$

Theorem 4. *A ruled surface $\Omega(s,v) = \zeta(s) + v\Gamma(s)$ is singular at the point (s_0, v_0) iff $\zeta'(s)$ and $\Gamma(s)$ are linearly dependent and*

$$v = 0.$$

Proof. Considering (24), there is the following equation

$$\zeta' \wedge \Gamma + v(\ell\Gamma - \omega) = 0$$

at the singular points of the ruled surface $\Omega(s,v)$. Here, there are two cases.

– The first case: $\zeta'(s)$ and $\Gamma(s)$ are linearly dependent and $\omega - \ell\Gamma = 0$. Considering (21), we can get the following

$$\omega - \ell\Gamma = \Gamma \wedge \Gamma' = 0 \tag{27}$$

Thus, $\alpha = \ell$, $\beta = 0$ and $\varepsilon = 0$. So, $\kappa = \sqrt{\varepsilon^2 + \beta^2} = 0$. Namely, ζ is a geodesic curve. However, we initially selected the curve ζ as a non-geodesic curve. So, this is not the case.

– The second case: $\zeta'(s)$ and $\Gamma(s)$ are linearly dependent and $v = 0$. This proof is complete.

According to parameter curves, the tangent vector fields of the ruled surface $\Omega(s,v) = \zeta(s) + v\Gamma(s)$ are

$$\Omega_s = \zeta' + v\Gamma', \quad \Omega_v = \Gamma.$$

The first basic form I of Ω is a Riemannian metric on Ω, and defined by

$$\mathrm{I} = E\mathrm{d}s^2 + 2F\mathrm{d}s\mathrm{d}v + G\mathrm{d}v^2.$$

The coefficient functions of this form are

$$E = g(\Omega_s, \Omega_s), \quad F = g(\Omega_s, \Psi_v), \quad G = g(\Omega_v, \Omega_v).$$

The second basic form II of Ω is

$$\mathrm{II} = l\mathrm{d}s^2 + 2m\mathrm{d}s\mathrm{d}v + n\mathrm{d}v^2,$$

and the coefficient functions are

$$l = g(\nabla_{\Omega_s}\Omega_s, N), \quad m = g(\nabla_{\Omega_s}\Omega_v, N), \quad n = g(\nabla_{\Omega_v}\Omega_v, N),$$

where, $\mathbf{N}$ is the unit normal vector field of the ruled surface $\Omega(s,v)$, so

$$\mathbf{N} = \frac{\Omega_s \wedge \Omega_v}{\|\Omega_s \wedge \Psi_v\|} = \frac{\zeta' \wedge \Gamma + v(\ell\Gamma - \omega)}{D}, \tag{28}$$

where, $D^2 = EG - F^2$. Thus, Gauss curvature K and the mean curvature H are

$$K = \frac{ln - m^2}{EG - F^2}, \tag{29}$$

$$H = \frac{1}{2}\frac{Gl - 2Fm + En}{EG - F^2}, \tag{30}$$

respectively.

4.1 Special Cases

Let M be a normal a.c.m. manifold and a curve ζ be a non-geodesic almost contact curve by the arc length s. We can examine the special cases of the ruled surface $\Omega(s,v) = \zeta(s) + v\Gamma(s)$ as follows:

4.2 The case where $\Gamma = \zeta'$

If $\Gamma = \zeta'$, then $\ell = g(\omega,\Gamma) = \alpha$, $\gamma_1 = 1$ and $\gamma_2 = \gamma_3 = 0$. Thus, $\alpha - \gamma_1 \ell = 0$. According to (25),

$$P_x = 0. \tag{31}$$

According to Teorem 3 and (31) , we can express the following corollaries.

Corollary 5. *$\Omega(s,v) = \zeta(s) + v\zeta'(s)$ is always developable.*

Corollary 6. *If (15) is used, in that case, the striction line of $\Omega(s,v) = \zeta(s) + v\zeta'(s)$ is*

$$\bar{\zeta}(s) = \zeta(s).$$

4.3 The case where $\Gamma = \varphi\zeta'$

If $\Gamma = \varphi\zeta'$, then $\ell = g(\omega,\Gamma) = \beta$, $\gamma_2 = 1$ and $\gamma_1 = \gamma_3 = 0$. According to (25),

$$P_x = \frac{\alpha}{\varepsilon^2 + \alpha^2}. \tag{32}$$

According to Teorem 3 and (32), we can express the following corollaries.

Corollary 7. *Let M be an cosymplectic or $\beta-$Kenmotsu manifold. $\Omega(s,v) = \zeta(s) + v\varphi\zeta'$ is always developable.*

Corollary 8. *Let M be a quasi-Sasakian manifold or $\alpha-$Sasakian manifold. $\Omega(s,v) = \zeta(s) + v\varphi\zeta'$ is never developable.*

Corollary 9. *Using (15), the striction line of the ruled surface $\Omega(s,v) = \zeta(s) + v\varphi\zeta'$ is*

$$\bar{\zeta}(s) = \zeta(s) - \frac{\varepsilon^2}{\sqrt{\varepsilon^2 + \alpha^2}}\varphi\zeta'(s).$$

4.4 The case where $\Gamma = \xi$

If $\Gamma = \xi$, then $\ell = g(\omega,\Gamma) = \varepsilon$, $\gamma_3 = 1$ and $\gamma_1 = \gamma_2 = 0$. According to (25),

$$P_x = \frac{\alpha}{\beta^2 + \alpha^2}, \tag{33}$$

According to Teorem 3 and (33), we can express the following corollaries.

Corollary 10.*Let M be a cosymplectic or $\beta-$Kenmotsu manifold. $\Omega(s,v)=\zeta(s)+v\xi$ is always developable.*

Corollary 11.*Let M be a quasi$-$Sasakian manifold or $\alpha-$Sasakian manifold. $\Omega(s,v)=\zeta(s)+v\xi$ is never developable.*

Corollary 12.*If (15) is used, the striction line of $\Omega(s,v)=\zeta(s)+v\xi$ is*

$$\bar{\zeta}(s)=\zeta(s)-\frac{\beta^2}{\sqrt{\beta^2+\alpha^2}}\xi.$$

4.5 The case where $\Gamma=Sp\{\varphi\zeta',\xi\}$

If $\Gamma=Sp\{\varphi\zeta',\xi\}$, then $\gamma_1=0$ and $\ell=g(\omega,\Gamma)=\gamma_2\beta+\gamma_3\varepsilon$. Thus, according to (25),

$$P_x=\frac{\det(\zeta',\Gamma,\Gamma')}{\|\Gamma'(s)\|^2}=\frac{\alpha}{\|\Gamma'(s)\|^2}. \tag{34}$$

According to Teorem 3 and (34), we can express the following corollaries.

Corollary 13.*Let M be a cosymplectic or $\beta-$Kenmotsu manifold. $\Omega(s,v)=\zeta(s)+v\Gamma(s)$ is always developable.*

Corollary 14.*Let M be a quasi$-$Sasakian manifold or $\alpha-$Sasakian manifold. $\Omega(s,v)=\zeta(s)+v\Gamma(s)$ is never developable.*

4.6 The case where $\Gamma=Sp\{\zeta',\xi\}$

If $\Gamma=Sp\{\zeta',\xi\}$, then $\gamma_2=0$, and $\ell=g(\omega,\Gamma)=\gamma_1\alpha+\gamma_3\varepsilon$. Thus, according to (25),

$$P_x=\frac{\det(\zeta',\Gamma,\Gamma')}{\|\Gamma'\|^2}=\frac{\alpha-\gamma_1\ell}{\|\Gamma'\|^2}=\frac{\left(1-\gamma_1^2\right)\alpha-\gamma_1\gamma_3\varepsilon}{\|\Gamma'\|^2}. \tag{35}$$

Theorem 5.*In the case $\Gamma=Sp\{\zeta',\xi\}$, $\Omega(s,v)=\zeta(s)+v\Gamma(s)$ is developable iff*

$$\frac{\varepsilon}{\alpha}=\frac{1-\gamma_1^2}{\gamma_1\gamma_3}=constant.$$

*Proof.*If we use (35), the proof is complete.

4.7 The case where $\Gamma=Sp\{\zeta',\varphi(\zeta')\}$

If $\Gamma=Sp\{\zeta',\varphi\zeta'\}$, then $\gamma_3=0$ and $\ell=g(\omega,\Gamma)=\gamma_1\alpha+\gamma_2\beta$. Thus, according to (25),

$$P_x=\frac{\det(\zeta',\Gamma,\Gamma')}{\|\Gamma'\|^2}=\frac{\alpha-(\gamma_1\alpha+\gamma_2\beta)\gamma_1}{\|\Gamma'\|^2}=\frac{\left(1-\gamma_1^2\right)\alpha-\gamma_1\gamma_2\beta}{\|\Gamma'\|^2}. \tag{36}$$

Theorem 6.*In the case $\Gamma=Sp\{\zeta',\varphi(\zeta')\}$, $\Omega(s,v)=\zeta(s)+v\Gamma(s)$ is developable iff*

$$\frac{\beta}{\alpha}=\frac{1-\gamma_1^2}{\gamma_1\gamma_2}=constant.$$

*Proof.*If we use (36), the proof is complete.

Theorem 7. *If the base curve ζ on $\Omega(s,v) = \zeta(s) + v\Gamma(s)$ is asymptotic, then*

$$\varepsilon(\gamma_3 + v\ell\gamma_2) + \beta(\gamma_2 + v\ell\gamma_3) = 0.$$

Proof. If the base curve ζ on the ruled surface $\Omega(s,v) = \zeta(s) + v\Gamma(s)$ is asymptotic, then

$$g\left(\nabla_{\zeta'}\zeta', \mathbf{N}\right) = 0, \tag{37}$$

where, $\mathbf{N}$ is the unit normal vector field of the surface $\Omega(s,v)$). If we substitute (11)) and (28) into (37)), the proof is complete.

Theorem 8. *If the base curve ζ on $\Omega(s,v) = \zeta(s) + v\Gamma(s)$ is geodesic, $\Omega(s,v)$ is always developable and the curvature function κ of the curve ζ satisfies the following equation*

$$\kappa^2 = \frac{\varepsilon(\gamma_2 + \gamma_3 v\ell) - \beta(\gamma_3 + \gamma_2 v\ell)}{v},$$

where $v \neq 0$.

Proof. If the base curve ζ on $\Omega(s,v) = \zeta(s) + v\Gamma(s)$ is geodesic, then

$$\nabla_{\zeta'}\zeta' \wedge \mathbf{N} = \mathbf{0}. \tag{38}$$

If we substitute (11) and (28) into (38), we obtain

$$\begin{aligned} 0 = {} & \left(-v\kappa^2 + \varepsilon(\gamma_2 + \gamma_3 v\ell) - \beta(\gamma_3 + \gamma_2 v\ell)\right)\zeta' \\ & + \beta v(\alpha - \gamma_1\ell)\varphi\left(\zeta'\right) + \varepsilon v(\alpha - \gamma_1\ell)\xi. \end{aligned}$$

Since the vectors $\{\zeta', \varphi\zeta', \xi\}$ are linearly independent, there are the following equations:

$$\begin{aligned} -v\kappa^2 + \varepsilon(\gamma_2 + \gamma_3 v\ell) - \beta(\gamma_3 + \gamma_2 v\ell) &= 0, \\ \beta v(\alpha - \gamma_1\ell) &= 0, \\ \varepsilon v(\alpha - \gamma_1\ell) &= 0. \end{aligned}$$

As shown in (26), $\alpha - \gamma_1\ell = 0$ is the criteria for developability, thus the proof is complete.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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