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Original new fixed point theorems in nth order $G-$ metric spaces

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Abstract: Many generalizations of the traditional metric space have been introduced in the literature, such as $2-$, $D-$, $G-$, $S-$ and $b-$metric spaces. When the studies on these generalized metric spaces are examined, it is seen that the main motivation of the researchers is to develop and generalize the famous Banach fixed point theorem. Although introduced with a similar motivation, its ability to measure the distance between n points simultaneously distinguishes the n th order $G-$metric space from other generalized metric spaces. In this study, we will give new and original fixed point theorems that reveal the importance of $G-$metric techniques since they cannot be reduced to the framework of quasi and conventional metric spaces.

Keywords: $G-$metric spaces, $G-$metric spaces with order n

1 Introduction

The famous Banach's fixed-point (for brevity FP) principle, based on the Polish mathematician Stefan Banach, is one of the most important results in metric fixed-point theory (for brevity FPT) and is considered the starting point of metric FPT. The main motivation of researchers working in the field of metric FPT has been to develop and generalize this famous theorem for nearly a century. In this sense, various generalized metric spaces such as $2-$metric, $D-$metric, $G-$metric, $S-$metric, and $b-$metric spaces are introduced using axiomatic methodology. Since the topology of most of these generalized metric spaces is not Hausdorff, some FP theorems required trivial and unnecessary additional conditions to obtain the desired fixed-point results in these spaces (see [9], [10] for more details).

It can be desirable to measure the distance between more than two items in metric spaces, which are basically based on the idea of measuring the distance between two points or objects. In such a case, it would be wise to combine the binary distance values for all pairs of items into an aggregate measure (see [13], [14] for more details). In this sense, nth order $G-$metric spaces are introduced as one of the last generalizations in the literature of conventional (usual) metric spaces where the distance between more than two elements can be measured simultaneously (for more details [4], [11], [23]). A generalization of the G-metric space well known in fixed point theory, nth order G metric spaces are topologically equivalent to a conventional metric space (see Example 3 and 4). Therefore, studies on G_n-metric spaces may not seem topologically important, but it should be noted that G_n-metric spaces and traditional (conventional) metric spaces are isometrically different. Consequently, the geometries on these spaces are different. Since the concept of metric equivalence is geometrically stronger than the concept of topological equivalence, the studies on G_n-metric space is geometrically very important and valuable.

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In addition, recent studies on the concept of statistical convergence in G_n-metric spaces (see [6], [7], [12] for more details) indicate that studies on G_n-metric spaces are very important not only in terms of geometry but also in terms of analysis and function theory.

Considering the FP results in $G-$metric spaces, most of these results can be obtained/derived from the well-known Park FP theorem (see Theorem 2) or from the well-known FP theorems in conventional metric spaces (see [1], [2], [8], [18], [19], [24] for more details). In this sense, our main motivation in this study is to give new and original FP theorems in G_n-metric spaces that cannot be reduced to the framework of quasi and conventional metric spaces. Thus, we will demonstrate the importance and need for the use of G_n-metric techniques and features.

2 Notation and Preliminaries

In this section, we introduce some notations and basic definitions and concepts related to the n order $G-$metric space that will be used later.

Definition 1. *Let $M \neq \emptyset$ be a set and $G_n : \overbrace{M \times \cdots \times M}^{n-times} \longrightarrow [0,+\infty)$ be a function. Then, G_n is called a $G-$metric with order n on M, if it satisfies the following conditions:*

(g1) *(positive definiteness): for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in M$,*

$$G_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \Longleftrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n,$$

(g2) *(permutation invariancy): let $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ be permutation function, then*

$$G_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = G_n\left(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(n)}\right),$$

(g3) *(monotonicity): for all $(\alpha_1, \alpha_2, \ldots, \alpha_n), (\beta_1, \beta_2, \ldots, \beta_n) \in M^n$*

$$G_n(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq G_n(\beta_1, \beta_2, \ldots, \beta_n)$$

such that $\{\alpha_i : i = 1, \ldots, n\} \subsetneqq \{\beta_i : i = 1, \ldots, n\}$,

(g4) *(generalized triangle inequality):*

$$G_n(\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t) \leq G_n(\alpha_1, \ldots, \alpha_s, w, \ldots, w) + G_n(w, \ldots, w, \beta_1, \ldots, \beta_t)$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_s, \beta_1, \beta_2, \ldots, \beta_t, w \in M$ and $s,t \in \mathbb{N}$ with $s+t=n$.

The pair (M, G_n) is called a $G-$metric space with order n. Briefly, (M, G_n) is called a G_n-metric space (see Definition 2.1 in [4] for more details).

In the next part of our work, for the sake of brevity, "$G_n ms$" notation will be used instead of G_n-metric space.

Theorem 1. *[[4]] Let $K \neq \emptyset$ be set. Then, (K,d) is a G_2ms iff (K,d) is a usual metric space.*

Remark. For the sake of brevity, the following notations will be used in the next sections;

(i) $G_n(\alpha, \beta, \ldots, \beta)$ by $G_n(\alpha; \beta)$,

(ii) $G_n(\alpha,\beta,\ldots,\beta,\gamma)$ by $G_n(\alpha;\beta;\gamma)$ or $G_n(\alpha,\gamma;\beta)$,

(iii) $G_n\left(\underbrace{\alpha,\ldots,\alpha}_{s-\text{times}},\beta,w,\ldots,w\right)$ by $G_n([\alpha]^s,\beta;\mathbf{w})$.

We can give a few examples of G_n- metric spaces as follows;

Example 1. [*Diameter G_n-metric* in [4]] The function d is defined by

$$\begin{aligned} d:\ \mathbb{R}^+\times\cdots\times\mathbb{R}^+ &\longrightarrow [0,+\infty) \\ (\alpha_1,\alpha_2,\ldots,\alpha_n) &\longmapsto d(\alpha_1,\alpha_2,\ldots,\alpha_n)=\max_{0\le i\le n}\alpha_i-\min_{0\le j\le n}\alpha_j. \end{aligned}$$

Then, the function d is a G_n-metric on $\prod_{i=1}^{n}\mathbb{R}^+$ for all $\alpha_1,\alpha_2,\ldots,\alpha_n\in\mathbb{R}^+$.

Example 2. [*Average G_n-metric* in [4]] Let (M,d_1) be usual metric space, and the function $d_2:M^n\to[0,+\infty)$ be defined by

$$d_2(\alpha_1,\alpha_2,\ldots,\alpha_n)=\frac{1}{n^2}\sum_{i,j=1}^{n}d_1(\alpha_i,\alpha_j).$$

Then, d_2 is a G_n-metric on M^n for all $\alpha_1,\alpha_2,\ldots,\alpha_n\in M$.

Example 3. [*Maximum and Additive G_n-metric* in [20]] Let (K,d) be usual metric space. Then, the following functions are G_n-metric on K^n:

$$\begin{aligned} G_n^M: K\times\cdots\times K &\longrightarrow [0,+\infty) \\ (\alpha_1,\ldots,\alpha_n) &\longmapsto G_n^M(\alpha_1,\ldots,\alpha_n):=\max_{0\le i,j\le n}\left\{d(\alpha_i,\alpha_j)\right\} \end{aligned}$$

and

$$\begin{aligned} G_n^S: K\times\cdots\times K &\longrightarrow [0,+\infty) \\ (\alpha_1,\ldots,\alpha_n) &\longmapsto G_n^S(\alpha_1,\ldots,\alpha_n):=\sum_{i=1}^{n}d(\alpha_i,\alpha_{i+1}) \end{aligned}$$

such that $\alpha_{i+1}=\alpha_1$ for $i=n$.

Example 4. [Example 4.6 in [20]] Let (K,G_n) be a $G_n ms$. Then, the following functions are usual metric on K :

i) $d^S(\alpha,\beta):=G_n(\alpha;\beta)+G_n(\beta;\alpha)$

ii) $d^M(\alpha,\beta):=\max\{G_n(\alpha_1,\alpha_2,\ldots,\alpha_n):\alpha_i\in\{\alpha,\beta\},\ 1\le i\le n\}$

Also, from the definitions of metrics d^S and d^M, it is clear that these metrics are equivalent. So, they generate the same topology on K.

Example 5. Let (K,G_n) be a $G_n ms$. We define the function d_{G_n} by

$$d_{G_n}(\alpha,\beta):=G_n(\alpha,\beta,\ldots,\beta).$$

Then, d_{G_n} is a quasi metric on K.

Lemma 1. *[Theorem 2.6 in [4], Lemma 4.1 in [20]] Let (M,G_n) be a $G_n ms$. Then, the following inequalities hold for all $x,y,w,x_1,x_2,\ldots,x_n\in M$.*

(i) $G_n([x]^s;\mathbf{w}) \leq sG_n(x;\mathbf{w})$.

(ii) $G_n([x]^s;\mathbf{w}) \leq (n-s)G_n(w;\mathbf{x})$.

Note that if we take $s=1$ in this last inequality, we will have the inequality $G_n(x,w,\ldots,w) \leq (n-1)G_n(w,x,\ldots,x)$, and this obtaining inequality will be used very often in the next part of our article.

(iii) $G_n(x_1,x_2,\ldots,x_n) \leq \sum_{i=1}^{n} G_n(x_i,w,\ldots,w)$.

(iv)

$$G_n(x_1;\mathbf{x_n}) \leq G_n(x_1;\mathbf{x_2}) + G_n(x_2;\mathbf{x_3}) + \cdots + G_n(x_{n-1};\mathbf{x_n})$$

or

$$G_n(x_k;\mathbf{x_1}) \leq G_n(x_2;\mathbf{x_1}) + G_n(x_3;\mathbf{x_2}) + \cdots + G_n(x_n;\mathbf{x_{n-1}}).$$

Definition 2. *[Multiplicity-Independent in [4]] Let (M,G_n) be a $G_n ms$. G_n-metric is called multiplicity-independent if the following condition holds*

$$G_n(\alpha_1,\alpha_2,\ldots,\alpha_n) = G_n(\beta_1,\beta_2,\ldots,\beta_n)$$

for all $(\alpha_1,\ldots,\alpha_n),(\beta_1,\ldots,\beta_n) \in M^n$ such that

$$\{\alpha_i : i=1,\ldots,n\} = \{\beta_i : i=1,\ldots,n\}.$$

Note that the concept of symmetry in G_3ms, which is known in the literature as $G-$metric space, corresponds to the concept of multiplicity independent in G_nms. Therefore, the concept of multiplicity independent is a more general concept that includes the concept of symmetry in $G-$metric space.

Remark.d^S has been defined in the Example 4 is reduced to

$$d^S(\alpha,\beta) = 2G_n(\alpha;\beta)$$

if G_n-metric is the multiplicity-independent. If G_n-metric is not multiplicity-independent the following inequalities hold;

$$\frac{n}{n-1}G_n(\alpha;\beta) \leq d^S(\alpha,\beta) \leq nG_n(\alpha;\beta).$$

Definition 3.*Let (K,G_n) be a G_nms and $\lambda \in K$ be a point. A sequence $(x_p)_{p\in\mathbb{N}}$ in K is said to be*

(i) *G_n-convergent to λ (shown as $(x_p) \overset{(K,G_n)}{\longrightarrow} \lambda$ or $(x_p) \to \lambda$) if, for any $\varepsilon \in \mathbb{R}^+$, there exists $i_0 \in \mathbb{N}$ satisfying $G_n\left(x_{i_1},\ldots,x_{i_{n-1}},\lambda\right) \leq \varepsilon$ for all $i_1,i_2,\ldots,i_{n-1}$ such that $i_1,i_2,\ldots,i_{n-1} \geq i_0$. That is,*

$$\lim_{i_1,\ldots,i_{n-1}\to+\infty} G_n\left(x_{i_1},\ldots,x_{i_{n-1}},\lambda\right) = 0.$$

(ii) *G_n-Cauchy if, for any $\varepsilon \in \mathbb{R}^+$, there exists $i_0 \in \mathbb{N}$ such that for all $i_1,\ldots,i_{n-1},i_n \geq i_0$, then $G_n\left(x_{i_1},x_{i_2},\ldots,x_{i_{n-1}},x_{i_n}\right) \leq \varepsilon$. That is,*

$$\lim_{i_1,\ldots,i_n\to+\infty} G_n\left(x_{i_1},\ldots,x_{i_{n-1}},x_{i_n}\right) = 0.$$

Lemma 2. *Let (K,G_n) be a G_nms and $(x_p)_{p\in\mathbb{N}}$ be a sequence in (K,G_n). We define the function d_{G_n} as in Example 5. Then, the sequence (x_p) is $G-$convergent to x iff the sequence (x_p) is $d_{G_n}-$convergent to x. Furthermore, (x_p) is G_n-Cauchy iff (x_p) is $d_{G_n}-$Cauchy.*

Definition 4. *Let (K,G_n) be a $G_n ms$. (K,G_n) is called G_n-complete metric space (for brevity $cG_n ms$) if every G_n-Cauchy sequence in (K,G_n) is G_n-convergent in (K,G_n).*

Definition 5. *Let (K,G_n) be a $G_n ms$ and $\alpha \in K$ and r >0. The set*

$$B_{G_n}(\alpha,r) = \{\beta \in M : G_n(\alpha,\beta,\ldots,\beta) < r\}$$

is called a G_n-ball with center α and radius r. Also, the family of all G_n-balls forms a base of a topology $\tau(G_n)$ on K, and $\tau(G_n)$ is called a G_n-metric topology.

Definition 6. *Let (K,G_n) and (K^*,G_n^*) be $G_n ms$.*

(i) *The map $g : K \longrightarrow K^*$ is said to be G_n-continuous at a point $x_0 \in K$ if $g^{-1}\left(B_{G_n^*}(gx_0,r)\right) \in \tau(G)$ for all $r > 0$.*
(ii) *The map $g : K \longrightarrow K^*$ is said to be G_n-continuous if it is G_n-continuous at all points of K.*
(iii) *The map $g : K \longrightarrow K^*$ is said to be G_n-homeomorphism if g is bijective, and g and g^{-1} are G_n-continuous.*

Lemma 3.*Let g be a map from a $G_n ms$ (M,G_n) to a $G_n ms$ (M^*,G_n^*). Then the following statements are equivalent*

(i) *g is G_n-continuous at $x \in M$*
(ii) *For all sequence $(x_p)_{p\in\mathbb{N}}$ in M such that $(x_p) \overset{(M,G_n)}{\longrightarrow} x$, $(gx_p) \overset{(M^*,G_n^*)}{\longrightarrow} gx$.*

Lemma 4.*Let (K,G_n) be a $G_n ms$, then $G_n(\alpha_1,\alpha_2,\ldots,\alpha_n)$ is jointly continuous function in all $n-$components of its variable.*

In view of Example 3 - 4, we can say that every $G_n ms$ is topologically equivalent to a metric space. So, it is quite meaningful to adopt concepts such as convergence, Cauchy sequence, continuity and completeness from metric spaces into the $G_n ms$ setting, since the G_n-metric topology $\tau(G_n)$ coincides with the metric topology arising from the metric d_{G_n}, d^S and d^M. In this sense, the prefix " G_n- " is written in front of the conventional concepts in the Definition 3 - 6 and Lemma 2, 3-4. It should be noted again that the counterparts of well-known concepts in $G_n ms$ will be written thanks to this front label. So, these concepts should never be considered as new concepts.

3 The fixed point theorems on G_n-metric spaces

3.1 From quasi metric to $G-$metric

In this section, generalizations of some important $G-$fixed point theorems in $G_n ms$ will be given. We show that these G_n-fixed point results can be deduced from Park fixed point theorem on quasi metric space. Similarly, it can be observed that many fixed point theorems on $G-$metric spaces in literature are particular case of Park fixed point theorem on quasi metric space (for more details [8], [19]). In the other words, although such fixed point theorems look like real and orginal generalizations, in fact they are not. So, G and G_n-metric FPT researches should be directed to fixed point results where the quasi and usual metric techniques are not useful and the Park result fails to be applicable. To this end, let us recall Park fixed point theorem:

Theorem 2. *[[21]] Let g be a self-map of a topological space (M,τ) and $d : M\times M \to [0,+\infty)$ be a lower semicontinuous such that $d(\alpha,\beta) = 0$ implies $\alpha = \beta$. If there exists $\alpha_o \in M$ such that $\lim_{p\to\infty} d\left(g^p\alpha_o, g^{p+1}\alpha_o\right) = 0$ and if α is a limit of a sequence $(g^p\alpha_o)_{p\in\mathbb{N}}$ with respect to τ. Finally, if $g : M \to M$ is orbitally continuous at α, then α is a fixed point of g.*

In the light of the Park fixed point theorem, we now give following fixed point theorem in complete G_n-metric spaces that can be proved without using $G-$metric techniques

Theorem 3. *Let (K,G_n) be a c$G_n$ms. Also, let $g:K\to K$ be mapping satisfying*

$$G_n(gx_1,gx_2,\ldots,gx_n)\le a\left[\sum_{i=1}^{n}G_n(x_i;\mathbf{gx_i})\right]\ \text{for all}\ x_1,\ldots,x_n\in K, \tag{3.1}$$

where $a\in\left[0,\frac{1}{n}\right)$. Then the mapping g has a unique fixed point. Furthermore, g is G_n-continuous at the fixed point.

Proof. Using the inequality (3.1), it is obtained that

$$G_n(gx;\mathbf{gy})\le a\left[G_n(x;\mathbf{gx})+(n-1)G_n(y;\mathbf{gy})\right].$$

From this inequality, the contraction condition in the following quasi metric framework is obtained;

$$d_{G_n}(gx,gy)\le a\left[d_{G_n}(x,gx)+(n-1)d_{G_n}(y,gy)\right] \tag{3.2}$$

for all $x,y\in K$.

Let's define the sequence $(x_p)_{p\in\mathbb{N}}\subset K$ such that $x_p=gx_{p-1}=g^px_o$ for the arbitrary point $x_0\in K$ with the help of Picard iteration. If $x_p=x_{p-1}$ for there exists $p\in\mathbb{N}$, then $x_{p-1}\in K$ is a fixed point of the mapping g. For all $p\in\mathbb{N}$ with $x_p\neq x_{p-1}$, using the inequality (3.2), we have that

$$d_{G_n}\left(g^px_o,g^{p+1}x_o\right)\le\lambda d_{G_n}\left(g^{p-1}x_o,g^px_o\right)$$

where $\lambda=\frac{a}{1-a(n-1)}$. Continuing this way, it is obtained that

$$d_{G_n}\left(g^px_o,g^{p+1}x_o\right)\le\lambda^p d_{G_n}(x_o,gx_o).$$

Hence, letting $p\to+\infty$ in above inequality, it is obtained that the sequence $(g^px_o)_{p\in\mathbb{N}}$ is an G_n-Cauchy sequence (see Lemma 2). The completeness of the G_n-metric space (K,G_n) implies that there exists $z\in K$ such that $(g^px_o)\to z$. Therefore, we have checked two of the required condition of Park fixed point theorem. It suffices to prove that g is orbitally continuous at z. Using the inequality (3.2), we have that

$$d_{G_n}\left(gz,g^{p+1}x_o\right)\le a\left[d_{G_n}(z,gz)+(n-1)d_{G_n}\left(g^px_o,g^{p+1}x_o\right)\right].$$

On letting $p\to+\infty$ in above inequality, we get that $gz=z$ which implies that $(g^px_o)\to z=gz$.

Consequently, we guarantees the existence of fixed point from Park fixed point theorem. To show the uniqueness of the fixed point, we assume that the mapping of g has two different fixed points u and v. Then, we have that

$$\begin{aligned}d_{G_n}(x,y)&=d_{G_n}(gx,gy)\\&\le a\left[G_n(x;\mathbf{gx})+(n-1)G_n(y;\mathbf{gy})\right]\\&\le 0,\end{aligned}$$

which implies that $d_{G_n}(x,y)=0$, that is, $u=v$.

Simillary, G_n-metric fixed point results in following Theorem 4 - 8 can be deduced from the quasi-metric framework. Avoiding repetition, we will not give proofs of these theorems. Note that these theorems are generalizations of many theorems in the $G-$metric fixed point literature (for instance, [15], [16], [17]).

Theorem 4. *Let (K,G_n) be a cG_nms. Also, let $g : K \to K$ be mapping satisfying, for all $x_1,\dots,x_n \in K$;*

$$G_n(gx_1,gx_2,\dots,gx_n) \le aG_n(x_1,x_2,\dots,x_n) + \sum_{i=1}^{n} b_i G_n(x_i;\mathbf{gx_i})$$

or

$$G_n(gx_1,gx_2,\dots,gx_n) \le aG_n(x_1,x_2,\dots,x_n) + \sum_{i=1}^{n} b_i G_n(gx_i;\mathbf{x_i})$$

where $\left(a+\sum_{i=1}^{n} b_i\right) \in [0,1)$. Then g has a unique fixed point. Furthermore, g is G_n-continuous at the fixed point.

Theorem 5. *Let (K,G_n) be a cG_nms. Also, let $g : K \to K$ be mapping satisfying, for all $x_1,\dots,x_n \in K$;*

$$G_n(gx_1,gx_2,\dots,gx_n) \le a \max_{1\le i\le n}\{G_n(x_i;\mathbf{gx_i})\}$$

or

$$G_n(gx_1,gx_2,\dots,gx_n) \le a \max_{1\le i\le n}\{G_n(gx_i;\mathbf{x_i})\}$$

where $a \in [0,1)$. Then g has a unique fixed point. Furthermore, g is G_n-continuous at the fixed point.

Theorem 6. *Let (K,G_n) be a cG_nms. Also, let $g : K \to K$ be mapping satisfying, for all $x_1,\dots,x_n \in K$;*

$$G_n(gx_1,gx_2,\dots,gx_n) \le aG_n(x_1,x_2,\dots,x_n) + b \max_{1\le i\le n}\{G_n(x_i;\mathbf{gx_i})\}$$

where $(a+b) \in [0,1)$. Then g has a unique fixed point. Furthermore, g is G_n-continuous at the fixed point.

Theorem 7. *Let (K,G_n) be a cG_nms. Also, let $g : K \to K$ be mapping satisfying, for all $x_1,\dots,x_n \in K$;*

$$G_n(gx_1,gx_2,\dots,gx_n) \le a \max_{1\le i,j\le n}\left\{G_n\left(x_i;\mathbf{gx_j}\right)\right\}$$

where $a \in \left[0,\frac{1}{2}\right)$. Then g has a unique fixed point. Furthermore, g is G_n-continuous at the fixed point.

Theorem 8. *Let (K,G_n) be a cG_nms. Also, let $g : K \to K$ be mapping satisfying, for all $x_1,\dots,x_n \in K$;*

$$G_n(gx_1,gx_2,\dots,gx_n) \le a \max_{1\le j\le n}\left\{\sum_{\substack{i\in\{1,\dots,n\}\\ i\ne j}}^{n} G_n\left(x_i;\mathbf{gx_j}\right)\right\}$$

where $a \in \left[0,\frac{1}{n-1}\right)$ and for $i=n$, $x_{n+1}=x_1$. Then g has a unique fixed point. Furthermore, g is G_n-continuous at the fixed point.

3.2 From usual metric to $G-$metric

In this section, we will give some FP theorems in $G_n ms$, which cannot be deduced from quasi metric framework. However, we have seen that such FP theorems counterpart the well-known celebrated FP theorems in usual metric space if $G_n ms$ is the multiplicity independent. In the case of not being multiplicity independent, it is expected that these fixed point theorems in $G_n ms$ cannot be deduced from the framework of quasi and usual metric spaces (see [18] for more details). So, the G_n-metric techniques become essential to prove the existence and uniqueness of these FP results. Consequently, the FP results in the non-multiplicity independent $G_n ms$ are real and original generalizations. The following Theorem 8 and Theorem 9 are instances of this kind of situation.

Note that indeed, for $a \in [1/2, 1)$, then the following fixed point theorem in $G_n ms$ cannot be deduced from quasi metric framework.

Theorem 9. *Let (K, G_n) be a $cG_n ms$. Also, let $g : K \to K$ be mapping satisfying, for all $x_1, \ldots, x_n \in K$;*

$$G_n(gx_1, gx_2, \ldots, gx_n) \leq a \max_{\substack{1 \leq i,j \leq n \\ i \neq j}} \left\{ G_n\left(x_i; \mathbf{gx_j}\right) \right\} \tag{3.3}$$

where $a \in [0, 1)$. Then g has a unique fixed point. Moreover, g is G_n-continuous at the fixed point.

Proof. Using the inequality (3.3), we have that for all $x, y \in K$

$$G_n(gx; \mathbf{gy}) \leq a \max\{G_n(x; \mathbf{gy}), G_n(y; \mathbf{gx})\}$$

and

$$G_n(gy; \mathbf{gx}) \leq a \max\{G_n(y; \mathbf{gx}), G_n(x; \mathbf{gy})\}.$$

From the sum of the two above inequalities, we have that

$$G_n(gx; \mathbf{gy}) + G_n(gy; \mathbf{gx}) \leq 2a \max\{G_n(x; \mathbf{gy}), G_n(y; \mathbf{gx})\}. \tag{3.4}$$

If (K, G_n) is the multiplicity independent, then we reduce the inequality (3.4) to the following inequality using definition of the metric d^S and Remark 2

$$d^S(gx, gy) \leq a \max\left\{d^S(x, gy), d^S(y, gx)\right\} \quad \text{for all } x, y \in K.$$

Since $0 \leq a < 1$, the existence and uniqueness of the fixed point of the map g is guaranteed by Bianchini fixed point theorem [3] in metric space $\left(K, d^S\right)$.

However, if (K, G_n) is non-multiplicity independent, then we similarly reduce the inequality (3.4) to the following inequality

$$d^S(gx, gy) \leq \frac{2a(n-1)}{n} \max\left\{d^S(x, gy), d^S(y, gx)\right\} \quad \text{for all } x, y \in K. \tag{3.5}$$

Since the coefficient $\frac{2a(n-1)}{n}$ is not always less than 1, the d^S contractive condition (3.5) will gives no information about the existence and uniqueness of the fixed point of the map g in metric space $\left(K, d^S\right)$. Similarly, within metrics equivalent to the d^S metric, it is not always guaranteed to be less than 1. So, the G_n-metric techniques, properties and methods become essential to prove the existence and uniqueness of the fixed point of the map g.

Let $x_0 \in K$ be an arbitrary point and the sequence $(x_p)_{p\in\mathbb{N}} \subset K$ be defined by $x_p = gx_{p-1} = g^p x_o$. If $x_p = x_{p-1}$ for some $p \in \mathbb{N}$, then $x_{p-1} \in K$ is a fixed point of the map g. So, for all $p \in \mathbb{N}$, let $x_p \neq x_{p-1}$, then using the inequality (3.3), we have that

$$\begin{aligned} G_n\left(x_p;\mathbf{x_{p+1}}\right) &= G_n\left(gx_{p-1};\mathbf{gx_p}\right) \\ &\leq a\max\left\{G_n\left(x_{p-1};\mathbf{gx_p}\right), G_n\left(x_p;\mathbf{gx_{p-1}}\right)\right\} \\ &= aG_n\left(x_{p-1};\mathbf{x_{p+1}}\right). \end{aligned}$$

By induction, we obtain

$$G_n\left(x_p;\mathbf{x_{p+1}}\right) \leq a^p G_n\left(x_0;\mathbf{x_1}\right).$$

Continuing this way, for all $r, p \in \mathbb{N}$ with $r > p$ we obtain

$$\begin{aligned} G_n\left(x_p;\mathbf{x_r}\right) &\leq G_n\left(x_p;\mathbf{gx_p}\right) + \cdots + G_n\left(x_{r-1};\mathbf{x_r}\right) \\ &\leq \left[a^p + \cdots + a^{r-1}\right] G_n\left(x_0;\mathbf{gx_o}\right) \\ &\leq \frac{a^p}{1-a} G_n\left(x_0;\mathbf{gx_o}\right). \end{aligned}$$

is obtained. Also, on taking limit as $p \to +\infty$ for above inequality, we have $G_n\left(g^p x_o;\mathbf{g^r x_o}\right) \to 0$. So, the sequence $(x_p)_{p\in\mathbb{N}} = (g^p x_o)_{p\in\mathbb{N}}$ is a G_n-Cauchy sequence. The completeness of the G_n-metric space (K, G_n) implies that there exists $u \in K$ such that $(g^p x_o) \to u$.

To show that $gu = u$, it will be sufficient to prove that $\left(g^{p+1}x_o\right) \to gu$. Suppose that $gu \neq u$. From the inequality (3.3), it is obtained that

$$G_n\left(x_{p+1};\mathbf{gu}\right) = G_n\left(gx_p;\mathbf{gu}\right) \leq a\max\left\{G_n\left(x_p;\mathbf{gu}\right), G_n\left(u;\mathbf{gx_p}\right)\right\}.$$

Taking limit as $p \to +\infty$, and using the fact that the function G_n is continuous in its variables, we get

$$G_n\left(u;\mathbf{gu}\right) \leq a\max\left\{G_n\left(u;\mathbf{gu}\right), G_n\left(u;\mathbf{gu}\right)\right\} = aG_n\left(u;\mathbf{gu}\right).$$

Since $a \in [0,1)$, this contradiction implies that $G_n\left(u;\mathbf{gu}\right) = 0$, that is, $gu = u$. Finally, we prove the uniqueness of the fixed point. Suppose that u, v be two distinct fixed points of the map g. Using the inequality (3.3), we have that

$$\begin{aligned} G_n\left(u;\mathbf{v}\right) &= G_n\left(gu;\mathbf{gv}\right) \\ &\leq a\max\left\{G_n\left(u;\mathbf{gv}\right), G_n\left(v;\mathbf{gu}\right)\right\} \\ &= a\max\left\{G_n\left(u;\mathbf{v}\right), G_n\left(v;\mathbf{u}\right)\right\}. \end{aligned}$$

But $a \in [0,1)$, the last inequality is reduced to

$$G_n\left(u;\mathbf{v}\right) \leq aG_n\left(v;\mathbf{u}\right) \tag{3.6}$$

Similarly, we have that

$$\begin{aligned} G_n\left(v;\mathbf{u}\right) &= G_n\left(gv;\mathbf{gu}\right) \\ &\leq a\max\left\{G_n\left(v;\mathbf{gu}\right), G_n\left(u;\mathbf{gv}\right)\right\} \\ &= a\max\left\{G_n\left(v;\mathbf{u}\right), G_n\left(u;\mathbf{v}\right)\right\}. \end{aligned}$$

Since $a \in [0,1)$, it is obtained that

$$G_n(v;\mathbf{u}) \leq aG_n(u;\mathbf{v}). \tag{3.7}$$

Considering the inequalities (3.6) and (3.7) together, we obtain that

$$G_n(u;\mathbf{v}) \leq aG_n(v;\mathbf{u}) \leq a^2 G_n(u;\mathbf{v})$$

Since $a \in [0,1)$, this contradiction implies that $u = v$.

Finally, to show that the map g is G_n-continuous at u, let $(y_p)_{p\in\mathbb{N}} \subset K$ be a sequence such that $\lim(y_p) = u$.

$$G_n(gy_p;\mathbf{u}) = G_n(gy_p;\mathbf{gu}) \leq a\max\left\{G_n(y_p;\mathbf{gu}), G_n\left(u;\mathbf{gy_p}\right)\right\}$$

On taking limit as $p \to +\infty$, we have $G_n(gy_p;\mathbf{u}) \to 0$ which implies that $gy_p \to u = gu$. By Lemma 3, the map g is the G_n-continuous at $u = gu$.

The following Theorem 10 is an original generalized FP theorem. Indeed, this theorem cannot be proved by reducing it to quasi metric space for $a \in \left(\frac{1}{4}, \frac{1}{2}\right)$. However, if $G_n ms$ is the multiplicity independent, then this FP theorem counterpart the FP theorem given by B. E. Rhoades [22] with the help of the contractive condition (22). In case of $G_n ms$ is not the multiplicity independent, in order for this theorem to be proved, techniques and properties specific to G_n-metric should be used. But here, although Theorem 11 cannot be proved by reducing to quasi metric spaces, we must immediately state that G_n-contractive condition required in the statement of Theorem 11 can be reduced to the contractive condition in Ciric fixed point theorem [5]. Therefore, we do not need G_n-metric techniques to ensure the existence and uniqueness of the fixed point in this theorem. In this sense, if G_n-contractive condition in G_n-fixed point theorems can be directly reduced to contractive condition in well known fixed point theorems on usual (or quasi) metric spaces, then these G_n-fixed point theorems can never be thought of as new and original fixed point theorems. However, this situation is overlooked by $G-$metric fixed point researchers.

Theorem 10. *Let (K, G_n) be a $cG_n ms$. Also, let $g : K \to K$ be mapping satisfying, for all $x_1, \ldots, x_n \in K$;*

$$G_n(gx_1, gx_2, \ldots, gx_n) \leq a \max_{1\leq i\leq n}\{G_n(x_i;\mathbf{gx_{i+1}}) + G_n(x_{i+1};\mathbf{gx_i})\}$$

where $a \in \left[0, \frac{1}{2}\right)$ and for $i = n$, $x_{n+1} = x_1$. Then g has a unique fixed point. Furthermore, g is G_n-continuous at the fixed point.

Theorem 11. *Let (K, G_n) be a $cG_n ms$. Also, let $g : K \to K$ be mapping satisfying, for all $x_1, \ldots, x_n \in K$;*

$$G_n(gx_1, \ldots, gx_n) \leq a \max_{1\leq i\leq n}\{G_n(x_1, \ldots, x_n), G_n(x_i;\mathbf{gx_i}), G_n(x_i;\mathbf{gx_{i+1}})\}$$

where $a \in \left[0, \frac{1}{2}\right)$ and for $i = n$, $x_{n+1} = x_1$. Then g has a unique fixed point. Furthermore, g is G_n-continuous at the fixed point.

4 Conclusion and Future Works

In the literature, it is a well-known fact by researchers working in this field that most of known FP results in Gms can be obtained from the quasi or usual metric space framework. It is controversial whether the $G-$FP results obtained without questioning whether they can be reduced to quasi and usual metric spaces are original and new results.

In this paper, we have demonstrated that many of G_n-FP theorems, which cannot be deduced from quasi metric framework, counterpart the well-known celebrated FP theorems in usual metric space if G_n-metric is the multiplicity-independent. In the case non-multiplicity independent, we give examples of the real and orginal FP theorems in G_nms are cannot be deduced from quasi and usual metric spaces framework.

In this context, we propose the following open problem:

Open Problem: If the metric space G_n, which is last generalization of usual and $G-$metric space in the literature, is multiplicity independent, under what geometrical and topological condition(s) can the fixed point theorems in this space be reduced to the quasi and usual space framework?

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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