New Trends in Mathematical Sciences

http://dx.doi.org/10.20852/ntmsci.2023.517

The Isometry Group of Tetrakis Hexahedron and Disdyakis Dodecahedron Spaces

Özcan Gelişgen[1] *and Zeynep Çolak*[2]

[1]Eskişehir Osmangazi University,Faculty of Sciences, Department of Mathematics and Computer Sciences, 26040, Eskişehir, Turkiye.
[2]Çanakkale Onsekiz Mart University, Biga Faculty of Economics and Administrative Sciences, Department of Business, Canakkale, Turkiye

Received: 8 August 2023, Accepted: 2 December 2023
Published online: 4 December 2023.

Abstract: Polyhedra are used in several fields by mathematicians and other scientists. It is easy to think of examples from architecture. Polyhedra have been used to scientifically explain the world around us. In the early days of study, polyhedra included only convex polyhedra. Since the ancient Greeks, many thinkers have worked on convex polyhedra. There are only five regular convex polyhedra known as Platonic solids, thirteen semi-regular convex polyhedra known as Archimedean solids, and thirteen irregular convex polyhedra which are duals of the Archimedean solids and known as Catalan solids. In this study, we show that the isometry group of the three-dimensional analytic space formed by the metrics of the Tetrakis hexahedron and the Disdyakis dodecahedron is the semi-direct product of O_h and $T(3)$, where the octahedral group O_h is the (Euclidean) symmetry group of the octahedron and $T(3)$ is the group of all translations of the three-dimensional space.

Keywords: Catalan solids, Metric, Isometry group, Octahedral Symmetry, Disdyakis Dodecahedron, Tetrakis hexahedron

1 Introduction

What is a polyhedron? Interestingly, this question is not easy to answer! "Traditional polyhedra" consist of plane faces, straight edges, and vertices. In the early days of study, polyhedra were only convex polyhedra. If the line segment connecting any two points in the set is also included in the set, the set is called a convex set. Since the ancient Greeks, many thinkers have studied convex polyhedra. The Greek scientists defined two classes of convex equilateral polyhedra with polyhedral symmetry, the Platonic and the Archimedean. Johannes Kepler found a third class, the rhombic polyhedra, and Catalan discovered a fourth class. The Archimedean solids and their twins, the Catalan solids, are less well known than the Platonic solids. While the Platonic solids consist of a single shape, these shapes that Archimedes wrote about consist of at least two distinct shapes, all forming identical vertices. There are thirteen polyhedra of this type. Since each solid has a "dual", there are also thirteen Catalan solids, named in 1865 after the Belgian mathematician Catalan. These are created by placing a point in the centre of the faces of the Archimedean solids and connecting the points with straight lines. The Catalan solids are all convex. They are surface-transitive, but not vertex-transitive. This is because the dual Archimedean solids are vertex-transitive and not face-transitive. Unlike the Platonic solids and the Archimedean solids, the faces of the Catalan solids are not regular polygons. However, the vertex figures of Catalan solids are regular, and they have constant dihedral angles.

Three main methods of geometric investigations: synthetic, metric and group approach. The group approach takes isometry groups of a geometry and convex sets plays an essential role in specifying the isometry group of geometries.

* Corresponding author e-mail: gelisgen@ogu.edu.tr

These properties are invariant under the group of motions and geometry studies these properties. One of the fundamental problems in geometry for S, which is a space with $d-$metric, is the definition of the group G of isometries. It is well known that G consists of transformations, rotations, reflections, slip reflections, and screws of the three-dimensional space when it is a Euclidean three-dimensional space with the usual metric. In this paper, we use the following descriptions cited by Martin [23]:

∘ A *transformation* is one to one equivalence from the set of points in space onto itself. If $d(X,Y) = d(\alpha(X), \alpha(Y))$ for every point X and Y, then α transformation is named an *isometry.*

∘ For all points X, if $\iota(X) = X$, then ι is called *identity.*

∘ If α fixes which set of points then α isometry is called a symmetry.

∘ For Δ plane, If $\sigma_\Delta(X) = X$ for point X on Δ and if $\sigma_\Delta(X) = Y$ for point X off Δ and Δ is perpendicular bisector of XY line segment, then σ_Δ, which is mapping on the points in $\mathbb{R}^3$, is called reflection.

∘ $\sigma_\Delta \sigma_\Gamma$ is defined *a rotation* about axis *l,* if Γ and Δ are two intersecting planes at line *l.*

∘ $\sigma_\Pi \sigma_\Delta \sigma_\Gamma$ is defined *a rotary reflection* about the common point to Γ, Δ and Π if Γ, Δ ,which each one perpendicular to Π, are intersecting planes.

∘ If $\sigma_N(X) = Y$ for every X points and N is midpoint of X and Y, then σ_N inversion about N is called *a transformation*. At the same time σ_N is defined *a point reflection.*

There are a lot of studies about group of isometries of a plane or a space (See [1,2,3,4,6,8,9,10,11,12,13,14,15,16,17, 18,19,21,22,24]). For example, the isometry group of taxicab, maximum and CC-space was showed in [12],[9],[13], respectively. In this work, we show that isometry group of the 3-dimensional analytical space furnished by tetrakis hexahedron and disdyakis dodecahedron metrics are the semi-direct product of O_h and $T(3)$, where octahedral group O_h is the (Euclidean) symmetry group of the octahedron and $T(3)$ is the group of all translations of the $3-$dimensional space.

2 Tetrakis Hexahedron, Disdyakis Dodecahedron and Preliminaries

The tetrakis hexahedron is a polyhedron with 24 faces, 36 edges, and 14 vertices. A tetrakis hexahedron (also called a tetrahexahedron, hextetrahedron, and kiskubus) is a Catalan solid (see Figure 1(a)). Its dual is the truncated octahedron, which is an Archimedean solid. It can also be called a disdyak hexahedron, since it is the dual of an omnitrunk tetrahedron. It can be thought of as a cube with square pyramids covering each square face that forms the top of the cube. Naturally occurring (crystal) formations of tetrahexahedra are observed in copper and fluorite systems. Polyhedral cubes in the form of the Tetrakis hexahedron are occasionally used by gamers[26] The Disdyakis dodecahedron is a polyhedron with 48 faces, 72 edges, and 26 vertices. The Disdyakis dodecahedron, also called hexakis octahedron or kisrhombic dodecahedron, is a Catalan solid and the dual of the Archimedean truncated cuboctahedron (see Figure 1(b)). As such, it is surface-transitive but has irregular surface polygons. The Disdyakis dodecahedron is the clover point of the rhombic dodecahedron[27]. We introduced the tetrakis hexahedron and disdyakis dodecahedron metrics in [15]. Let $\mathbb{R}^3_{TH}$ and $\mathbb{R}^3_{DD}$ which are called tetrakis hexahedron and disdyakis dodecahedron space denote 3-dimensional analytical space furnishing tetrakis hexahedron metric and disdyakis dodecahedron metric, respectively. The tetrakis hexahedron 3-dimesional space $\mathbb{R}^3_{TH}$ and the disdyakis dodecahedron 3-dimensional space are almost the same the Euclidean 3-dimesional space $\mathbb{R}^3$. The points, lines and planes are the same but the distance function is different. Both of them are the minkowski geometry.[See for detailed [25]]. The taxicab (Manhattan) and the maximum (Chebyshev) norms are defined as $\|X\|_1 = |x| + |y| + |z|$ and $\|X\|_\infty = \max\{|x|, |y|, |z|\}$, respectively and they are special cases of l_p-norm; $\|X\|_p = (|x|^p + |y|^p + |z|^p)^{1/p}$, where $X = (x,y,z) \in \mathbb{R}^3$. Among l_p-metrics only crystalline metrics, i.e., metrics having polygonal unit balls are l_1- and $l_\infty-$ metrics [7].

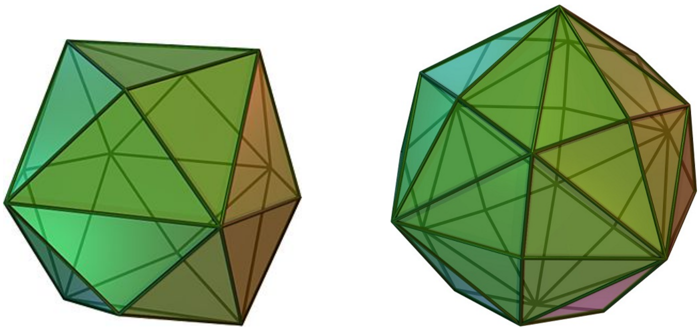

Fig. 1: (a) (b)

The tetrakis hexahedron and disdyakis dodecahedron metric and some properties of their are given briefly from [15]. First we give some notions that will be used in the descriptions of distance functions we define. For $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, M denotes $\|P_1 - P_2\|_\infty$ and S denotes $\|P_1 - P_2\|_1$. Moreover $X - Y - Z - X$ and $Z - Y - X - Z$ orientations are called positive (+) direction and negative (-) direction, respectively. M^+ and M^- expresses the next term in the respective direction according to M. For example, if $M = |x_1 - x_2|$, then $M^+ = |y_1 - y_2|$ and $M^- = |z_1 - z_2|$. The metrics for which the unit spheres are the tetrakis hexahedron and the disdyakis dodecahedron are defined as following:

Definition 1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. The distance function $d_{TH} : \mathbb{R}^3 x \mathbb{R}^3 \longrightarrow [0, \infty)$ tetrakis hexahedron distance between P_1 and P_2 is defined by*

$$d_{TH}(P_1, P_2) = \max\left\{M + \left(\sqrt{3} - 1\right) M^+ \ , \ M + \left(\sqrt{3} - 1\right) M^-\right\}$$

The distance function $d_{DD} : \mathbb{R}^3 x \mathbb{R}^3 \longrightarrow [0, \infty)$ disdyakis dodecahedron distance between P_1 and P_2 is defined by

$$d_{DD}(P_1, P_2) = \max\left\{M + \left(\sqrt{2} - 1\right) M^+ + \left(\sqrt{3} - \sqrt{2}\right) M^- \ , \ M + \left(\sqrt{2} - 1\right) M^- + \left(\sqrt{3} - \sqrt{2}\right) M^+\right\}$$

Let a, b, c indicate maximum one, middle one and minimum one of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively. Tetrakis hexahedron distance function and disdyakis dodecahedron distance function are

$$d_{TH}(P_1, P_2) = a + (\sqrt{3} - 1)b \qquad \text{and} \qquad d_{DD}(P_1, P_2) = a + (\sqrt{2} - 1)b + (\sqrt{3} - \sqrt{2})c,$$

respectively.

That is, d_{TH} distance is sum of maximum and ($\sqrt{3}$-1) times middle of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$ and d_{DD} distance is sum of maximum, ($\sqrt{2}$-1) times middle and ($\sqrt{3}$-$\sqrt{2}$) times minimum of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. Let l_1 be a line through P_1 and parallel to the coordinate axis which relates to the maximum of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. l_2, l_3 indicate lines each of which is parallel to a coordinate axis with respect to the middle and minimum of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively. Geometrically, the $TH-$path from P_1, P_2 is union of two line segments which one is parallel to l_1 and other line segments are made arctan($\frac{3-\sqrt{3}}{4}$) angle with l_2. Thus tetrakis hexahedron distance between P_1 and P_2 is the sum of Euclidean lengths of these two segments (Figure 2(a)). Geometrically, the $DD-$path from P_1 to P_2 is union of three line segments parallel to l_1 and other line segments are made $\frac{\pi}{4}$ and $arctan(\sqrt{2})$ angle with l_2 and l_3 respectively. Thus disdyakis dodecahedron distance between P_1 and P_2 is the sum of Euclidean lengths of these three line segments. Figure 2(b) shows the DD-way from P_1 to P_2 in the case $|y_1 - y_2| \geq |x_1 - x_2| \geq |z_1 - z_2|$. Thus, one can immediately state the following corollaries and lemma.

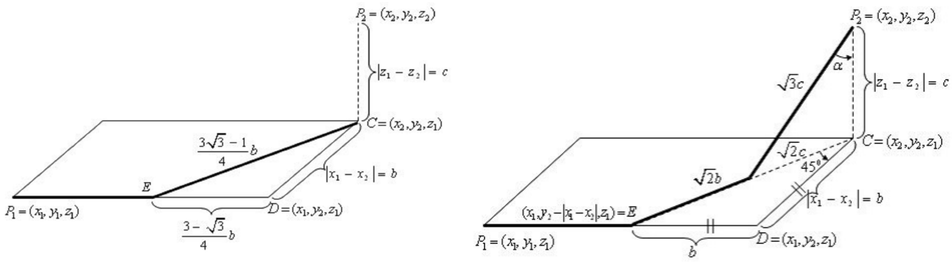

Fig. 2: (a) (b)

Corollary 1.*Let $M_0 = \|X - X_0\|_\infty$ for $X = (x,y,z)$ and $X_0 = (x_0,y_0,z_0)$. Equation of the tetrakis hexahedron with center (x_0,y_0,z_0) and radius r,*

$$\max\left\{M_0 + \left(\sqrt{3}-1\right)M_0^+ \ ,\ M_0 + \left(\sqrt{3}-1\right)M_0^-\right\} = r$$

which is a polyhedra which has 24-faces with vertices. Coordinates of the vertices are translations to (x_0,y_0,z_0) such that all permutations of the three axis components and all possible +/- sign changes of each axis component of $(0,0,r)$, and $(r/\sqrt{3}, r/\sqrt{3}, r/\sqrt{3})$(See Figure 3(a)).

Equation of the disdyakis dodecahedron with center (x_0,y_0,z_0) and radius r;

$$\max\left\{M_0 + \left(\sqrt{2}-1\right)M_0^+ + \left(\sqrt{3}-\sqrt{2}\right)M_0^- \ ,\ M_0 + \left(\sqrt{2}-1\right)M_0^- + \left(\sqrt{3}-\sqrt{2}\right)M_0^+\right\} = r$$

which is a polyhedra which has 24-faces with vertices. Coordinates of the vertices are translations to (x_0,y_0,z_0) such that all permutations of the three axis components and all possible +/- sign changes of each axis component of $(0,0,r)$, $(r/\sqrt{2}, r/\sqrt{2}, 0)$ and $(r/\sqrt{3}, r/\sqrt{3}, r/\sqrt{3})$(See Figure 3(b)).

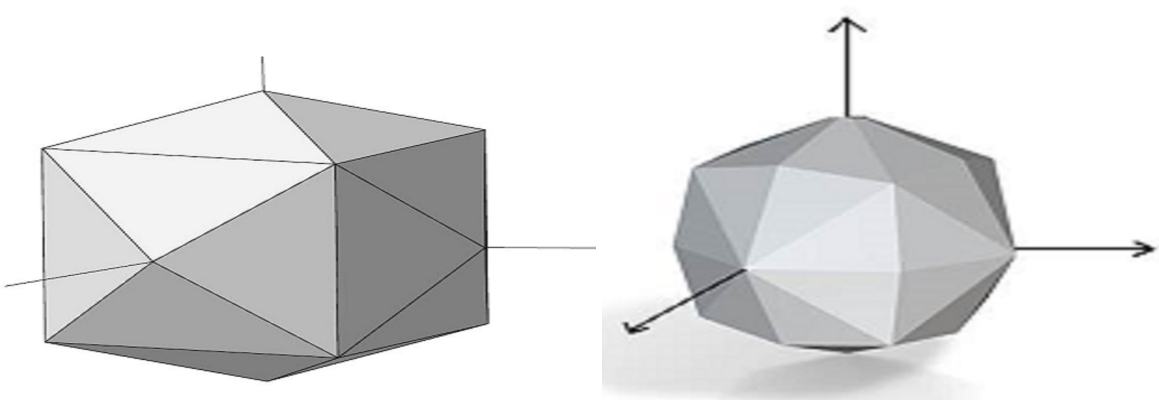

Fig. 3: (a) (b)

Lemma 1. *Let $M_d = \|P\|_\infty$ for $P = (p,q,r)$. Let l be the line through the points $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$ in $\mathbb{R}^3$ which is the analytical 3-dimensional space and d_E is Euclidean metric. If l has direction vector (p,q,r) then*

$$d_E(P_1,P_2) = \mu_{TH}(P_1P_2)d_{TH}(P_1,P_2) \text{ and } d_E(P_1,P_2) = \mu_{DD}(P_1P_2)d_{DD}(P_1,P_2)$$

where

$$\mu_{TH}(P_1P_2) = \frac{\max\left\{M_d+(\sqrt{3}-1)M_d^+ \;,\; M_d^+ +(\sqrt{3}-1)M_d^-\right\}}{\sqrt{p^2+q^2+r^2}},$$
$$\mu_{DD}(P_1P_2) = \frac{\max\left\{M_d+(\sqrt{2}-1)M_d^+ +(\sqrt{3}-\sqrt{2})M_d^- \;,\; M_d+(\sqrt{2}-1)M_d^- +(\sqrt{3}-\sqrt{2})M_d^+\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Above lemma tells us that d_{TH} and d_{DD} distance along any line is some positive constant multiple of Euclidean distance along same line. Thus one can get the following corollaries:

Corollary 2. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(X,P_1) = d_E(X,P_2)$ if and only if $d_{TH}(X,P_1) = d_{TH}(X,P_2)$ or $d_{DD}(X,P_1) = d_{DD}(X,P_2)$.*

Corollary 3. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then*

$$\frac{d_E(X,P_1)}{d_E(X,P_2)} = \frac{d_{TH}(X,P_1)}{d_{TH}(X,P_2)} = \frac{d_{DD}(X,P_1)}{d_{DD}(X,P_2)}.$$

Corollary 3 means that the ratios of Euclidean, d_{TH} and d_{DD} distances along a line are the same. In the following part of this article, we can study the isometries of $\mathbb{R}^3_{TH}$ and $\mathbb{R}^3_{DD}$, and describe their groups of isometries.

3 Isometries of The Tetrakis Hexahedron and Disdyakis Dodecahedron Spaces

The symmetry group of an object is the group of all transformations under which the object is invariant with composition as the group operation. For a space with a metric, the symmetry is a subgroup of the isometry group of the space concerned. Consider the octahedral group, also known as a cube group. This is the group of all self-isometries of that send a particular regular octahedron to itself. A regular octahedron has a symmetry order of 48 including transformations with reflection and rotation.

Tetrakis hexahedron and disdyakis dodecahedron have O_h octahedral symmetry. Its collective edges represent the reflection planes of the symmetry. It can also be seen in the corner and mid-edge triangulation of the regular cube and rhombic dodecahedron.

Three main methods for geometric investigations: synthetic, metric, and group approach. The group approach assumes isometry groups of a geometry, and convex sets play an essential role in specifying the isometry group of geometries. These properties are invariant under the group of motions and the geometry studies these properties. There are many studies on the isometry group of a space (see [1,2,3,4,6,8,9,10,11,12,13,14,15,16,17,18,19,21,22,24]). It was mentioned in the introduction that in Minkowski geometry the linear structure is the same as in Euclidean geometry, but the distance is not the same in all directions. Instead of the usual sphere in Euclidean space, the unit sphere is a certain symmetric closed convex set. In [20] the author gives the following theorem:

Theorem 1. *If the unit ball C of $(V,\|\|)$ does not intersect a two-plane in an ellipse, then the group $I(3)$ of isometries of $(V,\|\|)$ is isomorphic to the semi-direct product of the translation group $T(3)$ of $\mathbb{R}^3$ with a finite subgroup of the group of linear transformations with determinant ± 1.*

After this theorem remains a single question. This question is that what is the relevant subgroup?

Now we show that all isometries of the $\mathbb{R}^3_{TH}$ are in $T(3).G(TH)$, and also all isometries of the $\mathbb{R}^3_{DD}$ are in $T(3).G(DD)$. In the rest of article, we take $\triangle = TH$ or $\triangle = DD$. That is, $\triangle \in \{TH, DD\}$.

Definition 2. *Let P,Q be two points in $\mathbb{R}^3_{\triangle}$. The minimum distance set of P,Q is defined by*

$$\left\{X \mid d_{\triangle}(P,X) + d_{\triangle}(Q,X) = d_{\triangle}(P,Q)\right\}$$

and denoted by $[PQ]$ (See Figure 4).

In general, $[PQ]$ stand for a parallelepiped with diagonal PQ as in Figure 4(a)-(b).

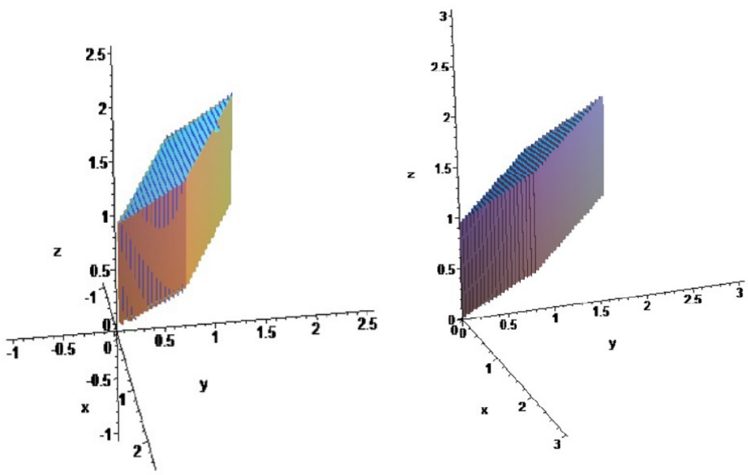

Fig. 4: (a) (b)

Proposition 1. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle}$ be an isometry and let $[PQ]$ be the parallelepiped. Then*

$$\phi([PQ]) = [\phi(P)\phi(Q)].$$

Proof. Let $Y \in \phi([PQ])$. Then,

$$\begin{aligned} Y \in \phi([PQ]) &\iff \exists X \in [PQ] \ni Y = \phi(X) \\ &\iff d_{\triangle}(P,X) + d_{\triangle}(Q,X) = d_{\triangle}(P,Q) \\ &\iff d_{\triangle}(\phi(P),\phi(X)) + d_{\triangle}(\phi(Q),\phi(X)) = d_{\triangle}(\phi(P),\phi(Q)) \\ &\Leftrightarrow \ Y = \phi(X) \in [\phi(P)\phi(Q)]. \end{aligned}$$

Corollary 4. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle}$ be an isometry and $[PQ]$ be the parallelepiped. Then ϕ maps vertices to vertices and preserves the lengths of the edges of $[PQ]$.*

Proposition 2. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle}$ be an isometry such that $\phi(O) = O$. Then $\phi \in G(\triangle)$.*

Proof. Since $\triangle \in \{TH, DD\}$, there are two possibility for $\triangle$. Let $\triangle = TH$, and let $A_0 = (0,0,1)$, $A_6 = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$, $A_{10} = (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ and $R = (0, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} + 1)$ be four points in $\mathbb{R}^3_{TH}$. Consider $[OR]$ that is the parallelepiped with

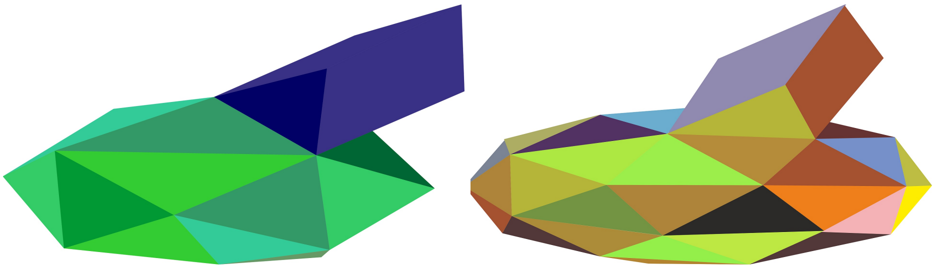

Fig. 5: (a) (b)

diagonal OR. (See Figure 5(a)) Also points A_0, A_6, A_{10} are on minimum distance set $[OR]$ and unit sphere with center at origin. Moreover these three points are the corner points of a tetrakis hexahedron's face which is a isosceles triangle. ϕ maps points A_0, A_6, A_{10} to the vertices of a tetrakis hexahedron by Corollary 4. Since ϕ preserve the lengths of the edges, $\phi(A_0) = A_i$, $\phi(A_6) = A_j$ and $\phi(A_{10}) = A_k$ such that $i \in \{0,1,2,3,4,5\}$ and $j,k \in \{6,7,8,9,10,11,12,13\}$. Since tetrakis hexahedron have 24 isosceles faces, there are 24 possibility to points which they can map, and also there are two possibility to points which they can map on the face of tetrakis hexahedron. Therefore total number of possibility are 48. Some of these cases can be seen as follows:

If $\phi(A_0) = A_0$, $\phi(A_6) = A_6$ and $\phi(A_{10}) = A_{10}$, then ϕ is the identity.

If $\phi(A_0) = A_1$, $\phi(A_6) = A_{13}$ and $\phi(A_{10}) = A_9$, then ϕ is the inversion.

If $\phi(A_0) = A_0$, $\phi(A_6) = A_{10}$ and $\phi(A_{10}) = A_6$, then $\phi = \sigma_\Delta$ is the reflection about the plane $\Delta : x = 0$

If $\phi(A_0) = A_0$, $\phi(A_6) = A_8$ and $\phi(A_{10}) = A_{12}$, then $\phi = \sigma_\Delta$ is the reflection about the plane $\Delta : y = 0$

If $\phi(A_0) = A_4$, $\phi(A_6) = A_7$ and $\phi(A_{10}) = A_{11}$, then $\phi = r_{\frac{3\pi}{2}}$ is the rotation with rotation axis $\parallel (1,0,0)$

The remaining cases can be similarly given.

Let $\triangle = DD$, and let $A_0 = (0,0,1)$, $A_{14} = (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $A_{18} = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ and $R = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{2}, \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{2} + 1)$ be four points in $\mathbb{R}^3_{DD}$. Consider $[OR]$ that is the parallelepiped with diagonal OR. (See Figure 5(b))

Also points A_0, A_{14}, A_{18} are on minimum distance set $[OR]$ and unit sphere with center at origin. Moreover these three points are the corner points of a disdyakis dodecahedron's face which is a scalene triangle. ϕ maps points A_0, A_{14}, A_{18} to the vertices of a disdyakis dodecahedron by Corollary 4. Since ϕ preserve the lengths of the edges, $\phi(A_0) = A_i$, $\phi(A_{14}) = A_j$ and $\phi(A_{18}) = A_k$ such that $i \in \{0,1,2,3,4,5\}$, $j \in \{6,...,17\}$ and $k \in \{18,...,25\}$. Since disdyakis dodecahedron have 48 scalene triangle faces, there are 48 possibility to points which they can map, and also there are only one possibility to points which they can map on the face of disdyakis dodecahedron. Therefore total number of possibility are 48. Some of these cases can be seen as follows:

If $\phi(A_0) = A_0$, $\phi(A_{14}) = A_{14}$ and $\phi(A_{18}) = A_{18}$, then ϕ is the identity.

If $\phi(A_0) = A_1$, $\phi(A_{14}) = A_{17}$ and $\phi(A_{18}) = A_{25}$, then ϕ is the inversion.

If $\phi(A_0) = A_0$, $\phi(A_{14}) = A_{14}$ and $\phi(A_{18}) = A_{22}$, then $\phi = \sigma_\Delta$ is the reflection about the plane $\Delta : x = 0$.

If $\phi(A_0) = A_5$, $\phi(A_{14}) = A_{16}$ and $\phi(A_{18}) = A_{20}$, then $\phi = r_{\frac{3\pi}{2}}$ is the rotation with rotation axis $\parallel (1,0,0)$.

If $\phi(A_0) = A_2$, $\phi(A_{14}) = A_6$ and $\phi(A_{18}) = A_{18}$ then $\phi = r_{\frac{4\pi}{3}}$ is the rotation with rotation axis $\parallel (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

The remaining cases can be similarly given.

Theorem 2.*Let* $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ *be an isometry. Then there exists a unique* $T_A \in T(3)$ *and* $\psi \in G(\triangle)$ *where* $\phi = T_A \circ \psi$

*Proof.*Let $\phi(O) = A$ such that $A = (a_1, a_2, a_3)$. ψ is definition of $\psi = T_{-A} \circ \phi$. We know that $\psi(O) = O$ and ψ is an isometry. Thereby, $\psi \in G(\triangle)$ and $\phi = T_A \circ \psi$ by Proposition 2. The proof of uniqueness is trivial.

4 Conclusion

In this paper, the spaces of which their sphere arethe Tetrakis hexahedron and the Disdyakis dodecahedron are introduced and some properties of metrics which are used setting up these space are given. Also, isometry groups of these spaces are given. So each of the groups of isometries of the spaces covering with the Tetrakis hexahedron and the Disdyakis dodecahedron is the semi-direct product of the icosahedral group O_h and $T(3)$, where O_h is the (Euclidean) symmetry group of the octahedron and $T(3)$ is the group of all translations of the 3-dimensional space. In the future works, handled solids in this paper are Catalan solids, the new metric space by considering different solids from these solids can be constructed and invastigate their some properties which are related to metrics.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Akça, Z. & Nazlı, S. (2022). *On the Shortest Distance in the Plane $R^2_{\pi/3}$*. New Trends in Mathematical Sciences, 10(4), 128?132.

[2] Can, Z., Gelişgen, Ö. & Kaya, R. (2015). *On the Metrics Induced by Icosidodecahedron and Rhombic Triacontahedron*. Scientific and Professional Journal of the Croatian Society for Geometry and Graphics (KoG), 19, 17–23.

[3] Bulgarean, V. (2013). *The Group $Isod_p(R^n)$ with $p \neq 2$*, Automation Computers Applied Mathematics Scientific Journal, 22(1), 69–74.

[4] Can, Z., Çolak, Z. & Gelişgen, Ö. (2015). *A Note On The Metrics Induced By Triakis Icosahedron And Disdyakis Triacontahedron*. Eurasian Academy of Sciences Eurasian Life Sciences Journal / Avrasya Fen Bilimleri Dergisi, 1, 1–11.

[5] Cromwell, P. R. (1997). Polyhedra, Cambridge University Press.

[6] Çolak, Z. & Gelişgen, Ö. (2015). *New Metrics for Deltoidal Hexacontahedron and Pentakis Dodecahedron*, SAU Fen Bilimleri Enstitüs ü Dergisi, 19(3), 353–360.

[7] Deza, M.M. & Deza, E. (2009). Encyclopedia of Distances. German: Springer, Berlin.

[8] Ekmekçi, S., Akça, Z., Altintas, A. & Bayar, A. (2016). Some Coplanar Points in Tetrahedron. Journal of Pure and Applied Mathematics: Advances and Applications, 16(2), 109-114.

[9] Ermiş, T. & Kaya, R. (2015). *On the Isometries the of 3- Dimensional Maximum Space*, Konuralp Journal of Mathematics, 3(1), 103–114.

[10] Ermiş, T., Savcı, Ü.Z. & Gelişgen, Ö. (2019). *A Note About Truncated Rhombicuboctahedron and Truncated Rhombicicosidodecahedron Space*, Sci. Stud. Res. Ser. Math. Inform., 29(1), 73-88.

[11] Ermiş, T. (2020) *Geometric Analysis of the Some Rectified Archimedean Solids Spaces via Their Isometry Groups*. Mathematical Sciences and Applications E-Notes, 8(2), 96–109.

[12] Gelişgen, O. & Kaya, R. (2009). *The Taxicab Space Group*, Acta Mathematica Hungarica, DOI:10.1007/s10474-008-8006-9, 122(1-2), 187–200.

[13] Gelisgen, Ö. & Kaya, R. (2015). *The Isometry Group of Chinese Checker Space*. International Electronic Journal Geometry, 8(2), 82–96.

[14] Gelişgen Ö. & Can, Z. (2016). *On The Family of Metrics for Some Platonic and Archimedean Polyhedra*. Konuralp Journal of Mathematics, 4(2), 25-33.

[15] Gelişgen Ö. & Çolak, Z. (2016). *A Family of Metrics for Some Polyhedra*. Automation Computers Applied Mathematics Scientific Journal, 25(1), 35–48.

[16] Gelişgen, Ö., Ermis, T. & Gunaltılı, I. (2017). *A Note About The Metrics Induced by Truncated Dodecahedron And Truncated Icosahedron*. InternationalJournal of Geometry, 6(2), 5-16.

[17] Gelisgen Ö. & Yavuz, S. (2019). *A Note About Isometry Groups of Chamfered Dodecahedron and Chamfered Icosahedron Spaces*. International Journal of Geometry, 8(2), 33–45.

[18] Gelisgen Ö. & Yavuz, S. (2019). *Isometry Groups of Chamfered Cube and Chamfered Octahedron Spaces*. Mathematical Sciences and Applications e-Notes, 7(2), 174–182.

[19] Gelisgen Ö. & Ermiş, T. (2020). *The Metris For Rhombicuboctahedron and Rhombicosidodecahedron*. Palestine Journal of Mathematics, 9(1), 15–25.

[20] Horvath, A. G. (2017). *Isometries of Minkowski geometries*. Lin. Algebra and Its Appl, 512, 172-190.

[21] Kaya, R., Gelisgen, O., Ekmekci, S. & Bayar, A. (2006). *Group of Isometries of CC-Plane*. Missouri J. of Math. Sci., 18(3), 221-233.

[22] Kaya, R., Gelisgen, O., Ekmekci, S. & Bayar, A. (2009). *On The Group of Isometries of The Plane with Generalized Absolute Value Metric*. Rocky Mountain Journal of Mathematics, 39(2), 591–603.

[23] Martin, G. E. (1997). Transformation Geometry. Springer-Verlag New York Inc.

[24] Savcı, Ü. Z. (2019). *Truncated Truncated Dodecahedron and Truncated Truncated Icosahedron Spaces*. Cumhuriyet Science Journal (CSJ), 40(2), 457-470.

[25] Thompson, A.C. (1996). Minkowski Geometry. Cambridge University Press.

[26] https://en.wikipedia.org/wiki/Tetrakis_hexahedron

[27] https://en.wikipedia.org/wiki/Disdyakis_dodecahedron