**Joint Laplace-Fourier Transforms For Fractional PDEs**

**A. Aghili \* , H. Zeinali**

Department of Applied Mathematics   
Faculty of Mathematical Sciences,  
University of Guilan.

P.O.Box, 1841, Rasht – Iran

[arman.aghili@gmail.com](mailto:arman.aghili@gmail.com)

[homa\_zeinaly@yahoo.com](mailto:homa_zeinaly@yahoo.com)

**Abstract**

In this paper, the authors implemented one dimensional Laplace transform to evaluate certain integrals, series and solve non homogeneous fractional PDEs. Illustrative examples are also provided. The results reveal that the integral transforms are very effective and convenient.

**Mathematics Subject Classification 2010**: 26A33; 34A08; 34K37; 35R11.

**Keywords**: Laplace transform; Bessel’s functions; Parabolic cylindrical function;

Wave equation; Caputo fractional derivative.

**1 Introduction and Notations**

In recent years, it has turned out that many phenomena in fluid mechanics, physics, biology, Engineering and other areas of sciences can be successfully modeled by the use of fractional derivatives. That is because of the fact that, a realistic modeling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus. Fractional differential equations arise in unification of diffusion and wave propagation phenomenon. The time fractional heat equation, which is a mathematical model of a wide range of important physical phenomena, is a partial differential equation obtained from the classical heat equation by replacing the first time derivative by a fractional derivative of order  In the last part of this paper we consider the time fractional wave equation (time fractional in the -Caputo sense).

In this work, we consider methods and results for the partial fractional diffusion equations which arise in applications. Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method , [ 1 ] ,[ 2 ] , [ 3 ] ,[ 5 ] the Fourier transform method [ 6 ], the iteration method and operational method [ 6 ].

**Definition.1.1.** Laplace transform of the function  is defined as follows



If , then  is given by



where  is analytic in the region .

**Example.1.2.** Evaluate Laplace transform of the parabolic cylindrical function



Solution. By definition we have



changing the order of integrals we get



the inner integral is Laplace transform of the function , so we can write the final result as following



Which can be written in the form



which can be evaluated by using partial method of integrating.

**Definition.1.3.** The left Caputo fractional derivative of order ( ) is defined as



**Lemma.1.4.** Let  be Laplace transform of the function  of exponential order with respect to ,then we have



In which 

Proof. We can write



where , on the other hand from Laplace transform table we know that



and also using the fact that (see [4])



in which . From relations (1.4), (1.5) and (1.6) we arrive at



in which and fractional derivation is considered in the Caputo sense. The final result will be obtained as below



Which can be re written in the form



**Definition.1.5.** Laguerre differential equation is defined as



which can be solved by using Laplace transform .Let us assume that



taking Laplace transform of Laguerre differential equation we obtain



**Lemma.1.6.** (Schouten-Vanderpol) Consider a function  which has the Laplace transform  which is analytic in the half plane . If  is also analytic for , then the inverse of  is as follows



Special case: 



Proof: See [6].

**Lemma.1.7**. Let X be an absolutely continuous random variable assuming non – negative

values,  its density and  its Laplace transform ( in such case  ( 

and  for real ).The knowledge of on the non – negative real line allows us to obtain some real moments of  through fractional integral and derivative of the - th order of . Many other expected values may be found from  or .

The following relations hold true



Proof : 1- By definition we have



which is equivalent to



2- We have



changing the order of integrals we get



3- Regarding the definition of Laplace transform we know that



changing the order of integrals and again using definition of Laplace transform the result will be obtained as following



4- By definition we have



changing the order of integrals we get



**Example.1.8.** Evaluate the following integral



Solution. Let , then regarding table of Laplace transform (see [8]) we have



one can prove that the function is a probability density function because



now by using first part of the previous lemma for , we can write



which can be evaluated by making a change of variable as below



**Definition.1.9.** The Fourier transform of the function  is defined as following



provided that the integral exists. The inverse of Fourier transform is



**Definition.1.10.** The finite Fourier sine transform of  in  is defined by



 where is an integer. The function  is then called the inverse finite Fourier sine transform of and is given by



 Similarly the finite Fourier cosine transform of  in  is defined by



where  is an integer. The function  is then called the inverse finite Fourier cosine transform of  and is given by



Lemma 1.11.The following relation holds true



Proof. We know the generating function of Legendre polynomials as below



substituting in the above relationship we will have



one can rewrite the above relationship as below



on the other hand using definition 1.5 we know that and



therefore if we take inverse Laplace transform of both sides of the above equation, we will have



**2 One dimensional Laplace transform of certain special functions**

The Bessel functions of the second kind, denoted by  or  are solutions of the Bessel differential equation that have a singularity at the origin . These are called Neumann or Weber functions as well. The Bessel functions are also valid for complex arguments , and an important special case is that of a purely imaginary argument. In this case, the solutions to the Bessel equation are called the modified Bessel functions (or occasionally the hyperbolic Bessel functions) of the first and second kind, and are defined by any of these equivalent alternatives



in which is Hankel function, and are Bessel functions of the first and second kind.

**Lemma.2.1.** The following relationship holds true



Proof. By the integral representation of modified Bessel function , we have



This leads to



By changing the order of integrals we have



It leads us to the following relationship



At this point, let us introduce a change of variables , we get



finally by using the fact that  and some easy calculations one gets

**Example.2.2.** Show that



Solution. It suffices to let  in lemma 2.1 to get the result.

**Lemma.2.3.** Assume , then we have the following integral representation



in special case , we have



Proof. See [5].

**Lemma.2.4.** The following relationship holds true



Proof. From lemma 2.3, we have



This leads to



Changing the order of integrals we get



Consequently we get the following relationship



by a change of variables , we have



finally by using the fact that  and some easy calculations we have

by some manipulations we get finally



Now let and to get the relationship ,it is provided that



and consequently it means that



**Lemma.2.5. (** Bobylev-Cercignani) Let F(p) be an analytic function having no singularities in the cut plane . Assume that  and the limiting value



exist for almost all

(i)  for and for ,uniformly in any sector 

(ii) there exists  such that for every 



where a(r) does not depend on  and  for any . Then, in the notation of the problem,



Proof. See [6].

**Example.2.6.** Using the previous lemma let  , one can check that  satisfies the conditions of lemma. Hence ,we may easily find the inverse of  by using the formula



substituting in the above formula leads to



making a change of variable  and using table of integrals, we have



**3 Main results**

The dynamic behavior of an overhead power wire which is connected to elecric locomotives by the panthograph can be simulated by a fractional wave partial differential equation which contains a term that shows the instant forces pushed towards the wire in certain moments.

**Problem.3.1.** Consider the following fractional PDE which describes the vibrations of an overhead wire under the power of an electric locomotive as a pantograph



under the following initial and boundary conditions



**Solution**. We solve this fractional PDE by using joint Laplace-Fourier finite sine transform. Taking Laplace transform of (6.1) with respect to t we have



now taking finite Fourier sine transform of the above relationship with respect to x we get



one can rewrite the above equation as below



By using lemma 1.6 we have



from (6.5), (6.6) and convolution theorem for Laplace transform we have



taking inverse finite Fourier sine transform, the result will be



**Problem.3.2.** Let us consider the following non homogenous time fractional PDE



**Solution**. ( Joint Laplace - Fourier transform ) Taking Laplace transform with respect to t of (6.7)

we have



in which . Now taking Fourier transform with respect to we have



or



and consequently





in which . Now invert with respect to  respectively. By using Schouten-Vander pol theorem we know that



also we know that



which can be rewritten as below



Now invert the above relation ( 6.10 ) with respect to . By using the definition of inverse Fourier transform we can write



which can be evaluated and simplified after change of integrals as following



or,



Finally, we get



In case of  one has simply



Where .

**4 Conclusion**

The paper is devoted to study applications of one dimensional Laplace transforms in details.

One dimensional Laplace transform provides a powerful method for analyzing linear systems. Certain time fractional wave equations with boundary conditions is solved. The method could lead to a promising approach for many applications in applied sciences.

**5 Acknowledgments**

The authors would like to thank the referees for their constructive comments and helps.

**References**

[1]A.Aghili, H.Zeinali,Integral transform method for solving Volterra Singular integral equations and non homogenous time Fractional PDEs**.** Gen.Math.Notes, Vol.14, No.1, January 2013, pp.6-20.

[2] A.Aghili, H.Zeinali, Integral transform methods for solving fractional PDEs and evaluation of certain integrals and series.Intern journal of physics and mathematical sciences, Vol.2(4),2012.

[3] V.A. Ditkin. and Prudnikov,A.P.: Operational Calculus In Two Variables and Its Application ,Pergamon Press, New York,1962.

[4] W.W.Bell, Special functions for scientists and engineers, D.Van Nostrand company LTD, Canada, 1968.

[5] D.G.Duffy, Transform methods for solving partial differential equations, Chapman and Hall/CRC NewYork,2004.

[6] H.J.Glaeske, A.P.Prudnikov, K.A.Skornik, Operational calculus and related topics, Chapman and Hall/CRC, USA, 2006.

[7] I. Podlubny, Fractional differential equations, Academic Press, San Diego, CA,1999.

[8] A.. D. Polyanin, A. V. Manzhirov, Handbook of integral equations, Chapman and Hall/CRC, USA, 2008.