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A note on Socle-Regular $QTAG$-Modules

Fahad Sikander · **Alveera Mehdi**

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Abstract A right module M over an associative ring with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. Recently the authors introduced the notions of socle regular and strongly socle-regular $QTAG$-modules and investigated their properties. Here we study those properties of these modules which are shared by their maximal h-divisible submodules and isometrically large submodules.

Keywords $QTAG$-modules, transitive modules, fully invariant modules.

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1 Introduction and Preliminaries

Throughout this paper, all rings will be associative with unity and modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M

Fahad Sikander
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
Tel.: +91-9457036190
E-mail: fahadsikander@gmail.com

Alveera Mehdi
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
E-mail: alveera_mehdi@rediffmail.com

is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height.

A submodule $N \subset M$ is nice [2, Definition 2.3] in M, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ, i.e. every coset of M modulo N may be represented by an element of the same height.

A fully invariant submodule $L \subset M$ is a large submodule of M, if $L + B = M$ for every basic submodule B of M. A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. For a limit ordinal α, $H_\alpha(M) = \bigcap_{\rho<\alpha} H_\rho(M)$, for all ordinals $\rho < \alpha$ and it is α-pure in M if $H_\sigma(N) = H_\sigma(M) \cap N$ for all ordinals $\sigma < \alpha$. A submodule $B \subseteq M$ is a basic submodule of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A characteristic submodule N of a $QTAG$-module M is a submodule that is invariant under each automorphism of M. For a submodule N of M, put $\sigma = \min\{H(x) \mid x \in \mathrm{Soc}(N)\}$ and denote $\sigma = inf(\mathrm{Soc}(N))$. Here $\mathrm{Soc}(N) \subseteq \mathrm{Soc}(H_\sigma(M))$. If K is submodule of M containing N, $inf(\mathrm{Soc}(N))$ may be calculated with respect to N and M respectively. To differentiate we write $inf(\mathrm{Soc}(N))_K$ and $inf(\mathrm{Soc}(N))_M$ respectively, but if K is an isotype submodule of M, then $inf(\mathrm{Soc}(N))_K = inf(\mathrm{Soc}(N))_M$. Several results which hold for TAG-modules also hold good for $QTAG$-modules [8]. Notations and terminology are follows from [6, 7].

2 Main Results

First we recall some basic definitions:

Definition 1 A h-reduced $QTAG$-module M is said to be *socle-regular* if for all fully invariant submodules N of M, there exists an ordinal σ such that Soc (N)=Soc $(H_\sigma(M))$. Hence σ depends on N.

Definition 2 A h-reduced $QTAG$-module M is said to be *strongly socle-regular* if for all characteristic submodules N of M, there exists an ordinal σ such that Soc (N)=Soc $(H_\sigma(M))$. Hence σ depends on N.

A strongly socle-regular $QTAG$-module is socle-regular but the converse is not true, in general. Here we investigate the conditions under which socle-regular modules become strongly socle-regular.

Proposition 1 *Let D be the maximal h-divisible submodule of a socle-regular (strongly socle-regular) $QTAG$-module M. Then M/D is also socle-regular (strongly socle-regular).*

Proof Let K be a fully invariant (characteristic) submodule of M such that N/K is also fully invariant (characteristic) submodule of M/K. For the endomorphism (automorphism) $f : M \to M$ we may define $\bar{f} : M/K \to M/K$ which is induced by f. Here $\bar{f}$ is well defined endomorphism (automorphism) of M/K. Since $\bar{f}(x+K) = f(x)+K$ and $\bar{f}(N/K) \subseteq N/K,\ f(N) \subseteq N$. Therefore N is fully invariant (characteristic) in M.

Now D is the maximal h-divisible submodule of M, it is fully invariant (characteristic), N is fully invariant (characteristic) in M whenever N/D is fully invariant (characteristic) in M/D. Therefore for some ordinal α,

$$Soc(N/D) = \frac{Soc(N)+D}{D} = \frac{Soc(H_\alpha(M))+D}{D} = \frac{Soc(H_\alpha(M)+D)+D}{D}$$

$$= Soc\left(\frac{H_\alpha(M)+D}{D}\right) = Soc(H_\alpha(M/D)).$$

This implies that M/D is socle-regular (strongly socle-regular).

Remark 1 If M is socle-regular (strongly socle-regular), then the maximal h-divisible submodule $D \subseteq M$, is also socle-regular (strongly socle-regular). Conversely, if $M = D \oplus A$, where A is h-reduced then for any fully invariant submodule C of M, $C = (C \cap D) \oplus (C \cap A)$ where $C \cap D$ is fully invariant in D and $C \cap A$ is fully invariant in A. If D and M/D are socle-regular, then M is also socle-regular as $M/D \cong A$.

Proposition 2 *Suppose that $M = N \oplus K$ with $H_\sigma(N) = H_\sigma(K)$ for some $\sigma \geq 0$.*

(i) *Then M is socle-regular if and only if M is strongly socle-regular, provided that $\sigma = n$ is an integer.*
(ii) *If M is fully transitive, then M is strongly socle-regular provided that $\sigma = \omega$.*

Proof (i) Let $M = N \oplus K$ with $H_n(N) = H_n(K)$, for some non-negative integer n. Then M is socle-regular if and only if M is strongly socle-regular [3]. Therefore (i) holds.

(ii) Let M be a $QTAG$-module with a decomposition $M = M_1 \oplus M_2$ such that $H_\omega(M_1)$ and $H_\omega(M_2)$ have the same Ulm supports. Then M is fully transitive if and only if M is transitive [3], therefore M should be transitive and we are done as every transitive $QTAG$-module is strongly socle-regular.

Definition 3 A large submodule L of of a $QTAG$-module M is said to be isometrically large if its every automorphism preserves heights, i.e., $H_M(x) \leq H_M(f(x)),\ \forall\ x \in L.$

It was proved in [4] and [5] that if L is a large submodule of the socle-regular (strongly socle-regular) $QTAG$-module M, then L is socle-regular (strongly socle-regular) as well.

For any $n \in \mathbb{N}$, $H_n(M)$ is isometrically large in M, but there may be isometrically large submodules which are not of the form $H_n(M)$. Also, a large submodule need not be isometrically large. Therefore we investigate these situations.

Proposition 3 *If L is isometrically large strongly socle-regular submodule of the $QTAG$-module M, then M is a strongly socle-regular $QTAG$-module.*

Proof Let N be a characteristic submodule of M. If $Soc(N) \nsubseteq H_\omega(M)$, then $\inf(Soc(N))$ is finite and by [5, Proposition 2.1], we have

$$Soc(N) = Soc(H_n(M))$$

for some non-negative n.

If $Soc(N) \subseteq H_\omega(M) = H_\omega(L)$, where the equality is well known, then $Soc(N)$ is a characteristic submodule of L by virtue of [3]. Thus there exists an ordinal $\sigma \geq 0$ with $Soc(N) = Soc(H_\sigma(L)) \subseteq Soc(H_\omega(L))$. So we may assume that $\sigma \geq \omega$ and as $H_\sigma(M) = H_\sigma(L)$, we have that $Soc(N) = Soc(H_\sigma(M))$ and we are done.

As an immediate consequence of the above, we have

Corollary 1 *If L is an isometrically large submodule of a $QTAG$-module M, then M is strongly socle-regular if and only if L is strongly socle-regular.*

We define fully nice complete and globally nice complete $QTAG$-modules as follows:

Definition 4 A QTAG-module M is said to be fully nice-complete if for every fully invariant submodule K of M, there exists a nice submodule N of M (eventually depending on K) such that $Soc(K) = Soc(N)$.

which states that each socle-regular $QTAG$-module is fully nice-complete. There is an immediate relation between socle-regular and fully nice-complete $QTAG$-modules

Definition 5 A $QTAG$-module M is said to be globally nice-complete if for every characteristic submodule Q of M, there exists a nice submodule N of M (eventually depending on Q) such that $Soc(Q) = Soc(N)$.

Similarly, strongly socle-regular $QTAG$-modules are globally nice-complete.

Evidently, strongly socle-regular $QTAG$-modules are globally nice-complete as well as globally nice-complete $QTAG$-modules are fully nice-complete.

We end this note with the following open problems:

Problem 1. If $M = L \oplus K$ with $H_\omega(L) = H_\omega(K)$ and M is socle-regular, does it follow that M is strongly socle-regular.

Problem 2. Characterize the classes of fully nice-complete $QTAG$-modules and globally nice-complete $QTAG$-modules.

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