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A new graph class defined by ferrers relation

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Abstract: In this study, we introduce a new graph class called Ferrers-esque coming from Ferrers relation. We give a characterization for the graph class by using the notion of distance in graphs.

Keywords: Distance in graphs, Algebraic combinatorics, Ferrers relation, Structural characterization of families of graphs.

1 Introduction and Preliminares

Ferrers relation by introduced for the first time [5] has been utilized for different purposes in extensive and various science fields. The relation was used with concept lattices in formal concept analysis [2]. Some graphs associated by the relation were linked together concept lattices again [6]. A kind of partition presented as Ferrers diagrams [1]. Some preference modelling structures were constructed by using a generalized version of the relation in Social Choice Theory [4]. In this study, we introduce a new graph class called Ferrers-esque graphs defined by the relation. We give a characterization of the class to be reasoned whether an arbitrary simple graph is in the class or not, in other words, the graph is a Ferrers-esque graph or not. Ferrers is called for Ferrers-esque graph for abbrivation.

Definition 1 [3] *A graph G is an ordered pair of disjoint sets (V,E) where $E \subseteq V \times V$. Set V is called the vertex or node set, while set E is the edge set of graph G. A simple graph does not contain self-loops.*

Definition 2 [7] *The distance d(u,v) between two vertices u and v of a finite graph is the minimum length of the paths connecting them. If no such path exists (if the vertices lie in different connected components), then the distance is set equal to ∞*

Definition 3 [5] *If a relation R over a set A is a Ferrers relation, it holds if aRb and cRd then either aRd or bRc for all distinct elements a, b, c, $d \in A$.*

Definition 4 *A simple graph $G = (V,E)$ is a Ferrers graph if for all distinct x, y, z, $w \in V$ if $xy \in E$ and $zw \in E$ then either $xw \in E$ or $yz \in E$. The definition of Ferrers graph must be extended to " if $xy \in E$ and $zw \in E$ then either $xw \in E$ or $yz \in E$ or $yw \in E$ or $xz \in E$ " since $xy \in E \Leftrightarrow yx \in E$ holds for all simple graphs.*

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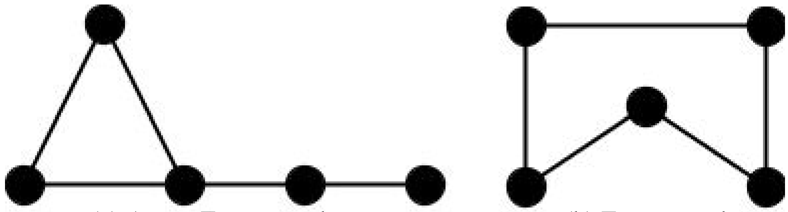

(a) A non-Ferrer graph (b) Ferrer graph

Fig. 1: A non-Ferrers and Ferrers graph example

2 Proofs

Theorem 5 *Let $G = (V,E)$ be a simple graph where $|V| \geq 4$. If G is a Ferrers then $d(v,w) \leq 2$ for all distinct v, $w \in V$.*

Proof. Let $G = (V,E)$ be a Ferrers but $n+1 = d(v,w) > 2$ for some $v,w \in V$ where $n > 2$. There exist at least $x_1, x_2, x_3, \ldots, x_n$ such that vx_1, $x_ix_{i+1}, \ldots, x_nw \in E$ where $n > 2$, $1 \leq i < n$ and if $j \neq i+1$ then x_ix_j, $x_jx_i \notin E$. Then the graph has a chain $vx_1, x_1x_2, \ldots, x_nw$ where $n > 2$. We have vx_1 and x_3w in E and G is Ferrers. Then vx_3 or vw or x_1x_3 or x_1w in E since G is Ferrers. If vw in E, then $d(u,w) = 1$ and this is a contradiction. If vx_3 then $d(v,x_3) = 1$ and $n+1 > d(v,w)$ so contradiction. This contradiction occurs when x_1x_3 or x_1w in E.

Theorem 6 *$G = (V,E)$ is a Ferrers if and only if for all dictinct $x,y,z,w \in V$ if $xy, zw \in E$ then $d(x,w) + d(y,z) \leq 4$*

Proof.($\Rightarrow$):
Let $G = (V,E)$ be a Ferrers graph but $d(x,w) + d(y,z) > 4$ for some distinct $x,y,z,w \in V$. If $xy, zw \in E$ for all distinct $x,y,z,w \in V$, $xw(wx)$ or $xz(zx)$ or $yz(zy)$ or $yw(wy)$ $\in E$.

Case 1. $xw(wx) \in E$:

If $xw \in E$ then we have a edge set at least $E_1 = \{xw, xy, zw\} \subseteq E$. Then $d(x,w) = 1$, $d(x,w) + d(y,z) > 4$. So $d(y,z) > 3$. We may take $d(y,z) = 4$. Then there exist x_1, x_2, x_3 such that yx_1, x_1x_2, x_2x_3, x_3z. it must holds yz or x_1x_3 or x_1z or yx_3 in E since G is a Ferrers and yx_1 and x_3z in E. But it is impossible to hold. So the graph is not a Ferrers graph. This causes a contradiction. Therefore, $d(x,w) + d(y,z) \leq 4$.

Case 2. $xz(zx) \in E$:

If $xz \in E$ then we have a edge set at least $E_2 = \{xz, xy, zw\} \subseteq E$. Then $d(x,z) = 1$, $d(x,y) = 1$, $d(z,w) = 1$. On the other hand, since $d(z,w) = 1$ and $d(z,x) = 1$ and $x \neq w$, $d(w,x) = 2$. $d(y,z)$ must be greater than 2 because we supposed that $d(x,w) + d(y,z) > 4$. We may assume $d(y,z) = 3$. Now, we can take yx_1, x_1x_2, x_3z in E. Suppose that $x = x_1$. Then $d(x,w) = 3$ and node x_2 must be between node w and x since $d(x,y) = 1$. However, $d(x,w)$ was equal to 2, so contradiction. Suppose that $x = x_2$. This means that $d(x,y) = 2$, so this is contradiction.

Case 3. $yz(zy) \in E$:

if $yz \in E$ then we have a edge set at least $E_3 = \{yz, xy, zw\} \subseteq E$ and $d(y,z) = 1$, $d(x,y) = 1$, $d(z,w) = 1$. It has to be held $d(x,w) > 3$ since $d(x,w) + d(y,z) > 4$. Suppose that $d(x,w) = 4$. Because $d(x,y) = 1$ and $d(y,z) = 1$ and also $x \neq z$, $d(x,z)$ must be equal to 1. Now, we have $d(z,w) = 1$ and $d(x,z) = 1$ and also $x \neq w$. So $d(x,w) = 1$. This contradict our assumption $d(x,w) = 4$.

Case 4. $yw(wy) \in E$:

If $yw \in E$ then we have a edge set at least $E_3 = \{yw, xy, zw\} \subseteq E$ and $d(y,w) = 1$, $d(x,y) = 1$, $d(z,w) = 1$. We obtain $d(y,z) = 1$ by using $d(y,w) = 1$ and $d(z,w) = 1$ and also $y \neq w$. It is true that $d(x,w) > 3$ since $d(x,w) + d(y,z) > 4$. Suppose that $d(x,w) = 4$. it must hold $d(x,w) = 2$ since $d(x,y) = 1$ and $d(y,w) = 1$ and also $x \neq w$. But it contradicts our assumptation $d(x,w) = 4$.

$(\Leftarrow)$:

Now, we prove that Let $G = (V,E)$ be a simple graph, if for all dictinct $x,y,z,w \in V$ "if $xy, zw \in E$ then $d(x,w) + d(y,z) \leq 4$", then $G = (V,E)$ is a Ferrers graph. Suppose that for all dictinct $x,y,z,w \in V$ "if $xy, zw \in E$ then $d(x,w) + d(y,z) \leq 4$ " but $G = (V,E)$ is not a Ferrers graph. If G is not a Ferrers, for some $x,y,z,w \in V$, $xy, zw \in E$ and yz, xw, yw, ,$xz \notin E$. On the other hand, if no path between two nodes v_1 and v_2, then $d(v_1, v_2) = \infty$ for any graph. We can not compute $d(x,w) + d(y,z)$ since $d(x,w) = \infty$ and $d(z,w) = \infty$. Therefore, if G is not a Ferrers, it is impossible to be held the inequality $d(x,w) + d(y,z) \leq 4$.

3 Conclusion

In this paper, we showed that a simple graph $G = (V,E)$ with $|V| \geq 4$ is a Ferrers graph if and only if for all distinct x, y, ,$z w$ $\in V$ if xy, $zw \in E$ then $d(x,w) + d(y,z) \leq 4$.

4 Future Work

Minimal Ferrers graph constructions and which mean obtaining Ferrers by adding minimal edges of numbers to a graph, efficient algorithms of the constructions will be studied on P_n, C_n and tree-like graphs. Next, domination numbers of minimal Ferrers constructions will be surveyed.

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