NEW TRENDS IN MATHEMATICAL SCIENCES
Vol. 1, No. 1, 2013, p.24-28
ISSN 2147-5520 - www.ntmsci.com

Numerical solving for nonlinear using higher order homotopy Taylor-perturbation

Nor Hanim Abd Rahman[1,*] , Arsmah Ibrahim[2], Mohd Idris Jayes[2]

[1]*Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 13500 Permatang Pauh, Pulau Pinang, Malaysia*

[2]*Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia*

Abstract: Rootfinding is a classical problem that still remains an interest to many researchers. A series of hybrid methods called Higher Order Homotopy Taylor-perturbation method via start-system functions (HTTPss) are implemented to give approximate solutions for nonlinear equations, $f(x) = 0$. The techniques serve as alternative methods for obtaining approximate solutions for different types of nonlinear equations. Thus, this paper presents an analysis on numerical comparison between the classical Newton Raphson (CNR), Homotopy Perturbation method (HTPss) and Higher Order Homotopy Taylor-perturbation via start-system (HHTPss). A computational system Maple14 is used for this paper. Numerical and Illustrative results reveal that HHTPss methods are acceptably accurate and applicable.

Keywords: HHTP, NONLINEAR, START SYSTEM

1. Introduction

Numerical method provides as an alternative way to solve nonlinear problems in many areas of science and technology. Among the numerical methods known for its efficiency and effectiveness is the Newton-Raphson method which is proven for its second order convergence. However there are problems in this method because its effectiveness lies in the accuracy and closeness of the initial value picked at the beginning of the iteration process.

There are also modified Newton methods for problem such as multiroots where a multiplier ‘m’ representing the highest power of higher order polynomial functions, is added into the original Newton-Raphson iterative method (S.G. Li et.al, 2010; Rafiq & Awais, 2008). However, it is effective only for simple nonlinear multiroots problems, but ineffective for more complex equations and higher order homotopy Taylor-perturbation. Besides the easiness of Newton-homotopy, it does not guarantee to converge. Other new and surprising methods also offer solutions to problems such as the perturbation technique that is based on an assumption that a small parameter must exist in the equation, the homotopy method (He, 1999, 2009), the hybrids such as homotopy perturbation (hpm), higher order hpm and hpm and with startsystem, as well as the robust approaches of these methods (Saeed et.al., 2011; Nor Hanim et.al, 2011a-d; Palancz, 2010; Saeed & Khthr, 2010; Pakdemirli & Boyaci, 2007). The applications of the start-system functions helps to accelerate the rate of convergence of the functions, because of the closeness of the start-system values suggested (Nor Hanim et.al, 2011 1-c). Their approaches are iterative, and in many cases appears to be considerably more computational oriented. The ideas of these new approaches are somehow simple and proven effective

***Corresponding author.** Tel.:+6-019-480-3797; fax:+0-000-000-0000.
E-mail address: norhanim@ppinang.uitm.edu.my.

This paper presents hybrid methods of higher order Taylor's series, perturbation techniques, homotopy continuation method and the start-system concepts in order to generate faster and more effective ways to solve multiroots of nonlinear problems for $f(x) = 0$.

2. Methodology

Higher-order homotopy Taylor-perturbation method involves the substitution of the perturbation techniques into the Taylors series up to the value of 'n' required, and the conversion of the original functions into homotopy functions (Nor Hanim et.al, 2011a-d, 2010). Furthermore, the convex homotopy for the function is defined as $H(x, \lambda): \Re \times [0,1] \rightarrow \Re$ as, $H(x, \lambda) = (1 - \lambda)p(x) + \lambda q(x) = 0$; where, λ is an embedded parameter, $\lambda \in [0,1]$; $p(x)$ as the start system function where $p(x) = x^n - C$, and n is preferably the highest power of x of a nonlinear function $f(x)$; $q(x)$ as the target system function; C is any real number in $f(x)$; and $H(x, 0) = p(x)$ & $H(x, 1) = q(x) = f(x)$. Here, the step size is set to 0.2 and the stop-criteria are set to $|x_{i+1} - x_i| < 10^{-5}$.

To determine the initial value x_0, only equate $p(x)$ to zero. The selection of $p(x)$ only requires a part of the original equation $f(x)$, which is known to have at least one trivial solution. There are also several other ways to identify a start-system of a linear homotopy as mentioned by Nor Hanim et.al. (2011 a-d) and Palancz et.al. (2010).

2.1. First Order Taylor-perturbation

The formulation of the first order Taylor-perturbation is illustrated. For a nonlinear equation, let $f(x) = 0$. Assume a perturbation expansion with only one correction term $x = x_0 + \varepsilon x_1$. By using algebraic manipulation $x - x_0 = \varepsilon x_1$, insert into Taylor Series expansion of order 1,

$$P_1(x) = f(x - x_0) \cong f(x_0) + f'(x_0)(x - x_0). \tag{1}$$

Substitute $\varepsilon x_1 = x - x_0$ into (1)

$$P_1(x) = f(\varepsilon x_1) \cong f(x_0) + f'(x_0)(\varepsilon x_1). \tag{2}$$

Assuming the RHS equals to 0 then becomes, and solve for εx_1, and we get

$$\begin{array}{c} f(x_0) + f'(x_0)(\varepsilon x_1) \cong 0 \\ \varepsilon x_1 = -f(x_0)/f'(x_0) \end{array} \rightarrow x = x_0 - f(x_0)/f'(x_0). \tag{3}$$

Add an iteration feature such as follow (equivalent to the classical Newton-Raphson),

$$x_{n+1} = x_n - f(x_n)/f'(x_n) \quad n = 0,1,2,3 \ldots \tag{4}$$

2.2. Second Order Taylor-perturbation (HTP)

The formulation second order Taylor-perturbation is illustrated. For a nonlinear equation, let $f(x) = 0$. Assume a perturbation expansion with two correction term $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2$, thus using algebraic manipulation we get,

$x - x_0 = \varepsilon x_1 + \varepsilon^2 x_2$. (5)

Inserting (5) into Taylor Series expansion of order 2,

$$P_2(x) = f(x - x_0) \cong f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2/2. \tag{6}$$

Substitute $\varepsilon x_1 + \varepsilon^2 x_2 = x - x_0$ into (6),

Similarly, expand and ignore the last 2 terms and factorize, we get

$$f(x_0) + f'(x_0)\varepsilon x_1 + f'(x_0)\varepsilon^2 x_2 + f''(x_0)\varepsilon^2 x_1^2/2 \cong 0, \tag{7}$$

$$f(x_0) + \varepsilon x_1 f'(x_0) + \varepsilon^2(x_2 f'(x_0) + x_1^2 f''(x_0)/2) \cong 0. \tag{8}$$

The sum of first 2 terms and the sum of last 2 terms of equation (8) be equal to 0,

$$f(x_0) + \varepsilon x_1 f'(x_0) \cong 0,$$
$$\varepsilon^2(x_2 f'(x_0) + x_1^2 f''(x_0)/2) \cong 0. \tag{9}$$

From equation 9, solve εx_1 and $\varepsilon^2 x_2$. Hence we get,

$$\varepsilon x_1 f'(x_0) = -f(x_0)$$
$$\therefore \varepsilon x_1 = -f(x_0)/f'(x_0); \tag{10}$$
$$\varepsilon^2 x_2 f'(x_0) + \varepsilon^2 x_1^2 f''(x_0)/2 \cong 0$$
$$\varepsilon^2 x_2 f'(x_0) = -\big(-f(x_0)/f'(x_0)\big)^2 f''(x_0)/2$$
$$\therefore \varepsilon^2 x_2 = -\big(f(x_0)\big)^2 f''(x_0)/2\big(f'(x_0)\big)^3; \tag{11}$$

Then, substitute equation (10) and (11) into equation (8),

$$x = x_0 - f(x_0)/f'(x_0) - f''(x_0)\big(f(x_0)\big)^2/2\big(f'(x_0)\big)^3. \tag{12}$$

Add an iteration feature such as follow, (The iteration scheme of HTP)

$$x_{n+1} = x_n - f(x_n)/f'(x_n) - f''(x_n)\big(f(x_n)\big)^2/2\big(f'(x_n)\big)^3;\ n = 0,1,2,3\ \ldots \tag{13}$$

Finally, convert equation (13) to homotopy equation,

$$x_{n+1} = x_n - H(x_n)/H'(x_n) - H''(x_n)\big(H(x_n)\big)^2/2\big(H'(x_n)\big)^3;\ n = 0,1,2,3\ \ldots \tag{14}$$

The same steps of calculations were done for order-3, order-4, order-5 and order-6. The equations were summarized as in Table 1. Hereafter, our discussion will only proceed with the above schemes. The derivations were first done manually. Then, countercheck using the mathematical software, *Maple14*.

Table1. The iteration scheme of the higher order correctional terms of homotopy Taylor-perturbation (HHTP) methods with start-system (ss).

Correctional Terms	**Higher Order Homotopy Taylor-Perturbation Method (HHTP-iterative form, $\boldsymbol{x = x_{(i+1)}}$)**
1st order	$x_i - H(x_i,\lambda)/H'(x_i,\lambda)$
2nd order	$x_i - [H(x_i,\lambda)/H'(x_i,\lambda)] - \big[H''(x_i,\lambda).H^2(x_i,\lambda)/2H'^3(x_i,\lambda)\big]$
3rd order	$x_i - [H(x_i,\lambda)/H'(x_i,\lambda)] - \big[H''(x_i,\lambda).H^2(x_i,\lambda)/2H'^3(x_i,\lambda)\big]$ $+ H^3(x_i,\lambda).\big[H'(x_i,\lambda)H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)/6H'^5(x_i,\lambda)\big]$
4st order	$x_i - [H(x_i,\lambda)/H'(x_i,\lambda)] - \big[H''(x_i,\lambda).H^2(x_i,\lambda)/2H'^3(x_i,\lambda)\big]$ $+ H^3(x_i,\lambda).\big[H'(x_i,\lambda)H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)/6H'^5(x_i,\lambda)\big]$ $+ \big[H''^2(x_i,\lambda).H^4(x_i,\lambda)/4.H'^6(x_i,\lambda)\big]$
5st order	$x_i - [H(x_i,\lambda)/H'(x_i,\lambda)] - \big[H''(x_i,\lambda).H^2(x_i,\lambda)/2H'^3(x_i,\lambda)\big]$ $+ H^3(x_i,\lambda).\big[H'(x_i,\lambda)H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)/6H'^5(x_i,\lambda)\big]$ $+ \big[H''^2(x_i,\lambda).H^4(x_i,\lambda)/4.H'^6(x_i,\lambda)\big]$ $- \big[H''^2(x_i,\lambda).H^5(x_i,\lambda).[H'''(x_i,\lambda) - 3.H'(x_i,\lambda)]/2.H'^8(x_i,\lambda)\big]$

6^{st} order	$x_i - [H(x_i,\lambda)/H'(x_i,\lambda)] - \left[H''(x_i,\lambda).H^2(x_i,\lambda)/2H'^3(x_i,\lambda)\right]$ $+ H^3(x_i,\lambda).\left[H'(x_i,\lambda)H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)/6H'^5(x_i,\lambda)\right]$ $+ \left[H''^2(x_i,\lambda).H^4(x_i,\lambda)/4.H'^6(x_i,\lambda)\right]$ $- \left[H''^2(x_i,\lambda).H^5(x_i,\lambda).[H'''(x_i,\lambda) - 3.H'(x_i,\lambda)]/2.H'^8(x_i,\lambda)\right]$ $+ H^2(x_i).H''(x_i)$ $/6.H'(x_i)^{10}.\left\{\left[(H'''(x_i).H''(x_i)^2H(x_i)^4/2)\right.\right.$ $\left.+ \left((H(x_i)/2H^2(x_i)^2) \times (H'(x_i).H'''(x_i) - 3H'(x_i)^2)\right)^2\right]$ $\left.+ \left[\left((3/4)f''(x_i)^3.f(x_i)^4\right)\right]\right\}$

Sources:(Nor Hanim et.al, 2011a-d)

3. Numerical Analysis

The list of the higher order iterative schemes of Homotopy Taylor-perturbation by using start-system (HHTPss) can be referred at Table 1. While, Table 2 shows the selected nonlinear equations, start-system functions, $p(x)$, and the selected initial value, x_0. The efficiency of the iterative hybrid methods from 1^{st} order to the 5^{th} order homotopy Taylor-perturbation method using start-system (HHTPss) is also given in Table 2, which gives equal or better results in terms of convergence rate as compared to the classical Newton-Raphson. The given test function (i)-(xii) are used and the results of the approximated zeros is given in 10^{-5} error accuracy. Furthermore, the choice of a suitable $p(x)$ is not unique and different choices of $p(x)$ work better for different types of equations.

Table 2: The approximated zeros using Classical Newton-Raphson (CNR) and higher order Homotopy Taylor-Perturbation (HHTP) via start-system: startsystem function, initial value and number of iterations needed to converge.

	NL Functions	**CNR**	**1^{st}**	**2^{nd}**	**3^{rd}**	**4^{th}**	**5^{th}**
	$q(x) = f(x)$; $p(x)$; $p(x) = 0 = x_0$						
i	$\sin(3x) + 4(e^{2x}) - e^{(-x)} + \ln(2x) - 3$ $\left(e^{(x)} + -3\right)$ 1098612289	5	2	2	1	4	5
ii	$\frac{1}{2} + \frac{1}{4}x^2 - \sin x - \cos(2x)$ (x^2) 0.5	3	1	2	1	1	1
iii	$x^2 - 4\tan(2x) - \tanh(x)$ $(\tanh(x))$; 0.0	4	1	1	1	1	1
iv	$(e^x + x^2 - 2)^4$ $(x^{16} - 16)$ -1.189207115,1.189207115	37	34	24	20	20	20
v	$(2x^2 - 9)^3.(\ln(x))$ $(x^2 - 9)$ -3.0,3.0	27	21	15	13	11	11
vi	$(\tan(x) - \tanh(x))^5$ $\tan(x)$ 0.0,-2.514866859, 2.514866859*	118	39	28	24	24	23
vii	$(x^5 - x^3 - 2)^3$ $(x^5 - 2)$ 1.148698355	21	19	14	12	12	12
viii	$(e^x - \sin(x))^2$ $(\sin(x))$; 0.0,-3.0*	14	12	9	8	8	8
ix	$(xe^x + 2^{-x} + 2\cos(x) - 6)^2$	3	1	1	1	1	1

	$(\cos(x))$ 1.570796327						
x	$(x-2)^2.(x^4+6x-40)$ (x^4-40); 2.514866859,2.514866859	6	5	3	3	3	3
xi	$(x-2)^2(\sin(x))$ (x^3-2); 1.259921050	4	2	2	2	2	2
xii	$(x^2-e^x-3x+2).(\cos(x))$ (x^2-3x);0.0, 3.0	3	2	1	1	1	1

4. Conclusions

It is concluded that the method of higher-order homotopy Taylor-perturbation is an effective alternative method in accelerating the converge rate in solving nonlinear functions. Combined with the method of start-system, it facilitates the way to determine the initial value for the iteration. Thus, the computing time is reduced.

Acknowledgement

We would like to thank Universiti Teknologi MARA Malaysia (UiTM) and Ministry of Higher Education Malaysia for the financial supports.

References

[1] Chun,C., Bae, H.J. & Neta, B. 2009. New families of nonlinear third-order solvers for finding multiple roots. *Computer and Mathematics with Applications* **58**:1574-1582.

[2] He, J-H. 1999. Homotopy perturbation technique. *Computer Methods in Applied Mechanic and Engineering* **178**:257-262.

[3] He, J-H. 2009. An introduction to the homotopy perturbation method. Computer and Mathematics with applications. **57(3)**:410-412. Doi:10.1016/j.camwa.2008.06.003

[4] Palancz, B., Awange, J.L & Lewis, R.H. 2010. Linear Homotopy Solution of Nonlinear Systems of Equations in Geodesy. *J. Geod*. Doi 10.1007/s100190-009-0346-x.

[5] Pakdemirli, M. & Boyaci, H. 2007. Generation of root finding algorithms via perturbation theory and some formulas. *Applied Mathematics and Computation,* **184**:783-788.

[6] Rafig, A & Awais, M. 2008. Convergence on the Homotopy Continuation Method. *International Journal of Applications Mathematics and Mechanics,* **4(6)**:62-70.

[7] Saeed, R.K. & Khthr, F.W. 2010. Three new iterative methods for solving nonlinear equations. *Australian Journal of Basic & Applied Sciences* **4(6):**1022-1030.

[8] S.G. Li, L.Z.Cheng & B.Neta. 2010. Some fourth-order nonlinear solvers with closed formulae for multiple roots. Computers and Mathematics with Applications **59**:126-135

[9] Nor Hanim Abd Rahman, Arsman Ibrahim, Mohd Idris Jayes. 2011d. Nor Hanim Abd Rahman, Arsmah Ibrahim & Mohd Idris Jayes (2011). Higher Order Homotopy Taylor-Perturbation using Start-system for Multifunction Problems. *Sains Malaysiana-SKSM19, special edition,* Publisher UKM; accepted (July2011)

[10] Nor Hanim Abd Rahman, Arsman Ibrahim, Mohd Idris Jayes. 2011a. Higher Order Homotopy Taylor-Perturbation using Start-system in Solving Nonlinear Equations. *Abstract Book: Regional Fundamental Science Congress 2011*. pp.351-352. Faculty of Science, UPM. 2011 ISBN: 978-983-2519-13-3

[11] Nor Hanim Abd Rahman, Arsman Ibrahim, Mohd Idris Jayes. 2011b. Solving Nonlinear Equations using Improved Higher Order Homotopy Perturbation with start-system. *(CD) Proceeding IceMATH2011: The International Conference on Numerical Analysis & Optimization.* Universitas Ahmad Dahlan, Yogyakarta, Indonesia (June 6-8, 2011). ISBN:978-602-98919-1-1

[12] Nor Hanim Abd Rahman, Arsman Ibrahim, Mohd Idris Jayes. 2011c. Newton-Homotopy solution for nonlinear equations using Maple14. *Journal of Science and Technology* **3(2):** 69. December 2011; ISSN 2229-8460