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An approach to numerical solutions of system of high-order linear differential-difference equations with variable coefficients and error estimation based on residual function

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Received: 27 September 2014, Revised: 19 October 2014, Accepted: 14 Novamber 2014
Published online: 6 December 2014

Abstract: In this study a method is presented which aims to make an approach by using Bernstein polynomials to solutions of systems of high order linear differential-difference equations with variable coefficients given under mixed conditions. The method converts a given system of differential-difference equations and the conditions belonging to this system to equations that can be expressed by matrices by using the collacation points and provides to find the unknown coefficients of approximate solutions sought in terms of Bernstein polynomials. Different examples are presented with the purpose to show the applicability and validity of the method. Absolute error values between exact and approximate solutions are computed. The estimated values of absolute errors are computed by using the residual function and these estimated errors are compared with absolute errors. For all numerical computations of this study the computer algebraic system Maple 15 is used.

1 Introduction

Differential-difference equations and systems of these class of equations are used to model various science and engineering problems. Since solving systems of differential-difference equations analytically is hard, several numerical methods are improved to solve these systems. We can give examples of some particular methods used for solving these systems such as variational iteration method [16], differential transformation method [1], Adomian decomposition method [9], differential transform method [17], linearizability criteria [10], decomposition method [4], homotopy analysis method [21], homotopy perturbation method [12], Bessel matrix method [20], Taylor collacation Method [8].

In this study, modifying and developing methods in [2,8,13,20] and using matrix relations between Bernstein polynomials and their derivatives, we present an approach to numerical solutions of systems of linear high-order differential-difference equations with variable coefficients in the form

$$\sum_{r=0}^{m}\sum_{i=1}^{k} P_{j,i}^{r}(x)y_i^{(r)}(\lambda x+\beta)=f_j(x), \quad j=1,2,\cdots,k \tag{1}$$

given together with mixed conditions defined as follows

$$\sum_{j=1}^{m-1} a_{r,j}^{n}(x)y_n^{(j)}(a)+b_{r,j}^{n}(x)y_n^{(j)}(b)+c_{r,j}^{n}(x)y_n^{(j)}(c)=\lambda_{n,r}, \tag{2}$$
$$a\le c\le b, \quad r=0,1,2,\cdots,m-1, \quad n=1,2,\cdots,k$$

where y_j represents an unknown function, $P_{j,i}$ represent the known functions and f_j are defined on the closed interval $[a,b]$ and $a_{r,j}$, $b_{r,j}$, $c_{r,j}$ and $\lambda_{n,r}$ are appropriate constants.

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Our main purpose is to find the approximate solutions of system given with (1) expressed in the following truncated Bernstein series form:

$$y_i(x)=\sum_{n=0}^{N} y_{n,i}B_{n,N}(x), \quad a\le x\le b, \quad i=1,2,\cdots,k \tag{3}$$

where $y_{n,i}$, $(i=1,2,\cdots,k,\ n=0,1,\cdots,N)$ are the unknowns coefficients to be determined and N is any positive integer such that $N\ge m$.

2 Bernstein Polynomials

General form of the nth degree Bernstein polynomials are defined by [3] as follows:

$$B_{i,N}(x)=\binom{N}{i}\frac{x^i(R-x)^{N-i}}{R^N}, \quad i=0,1,\cdots,N$$

where where R is the maximum range of the interval $[0,R]$ over which the polynomials are defined to form a complete basis. In [3] it is mentioned that there are $n+1$ nth degree polynomials and $B_{i,N}(x)=0$ for $i<0$ and $i>N$.
Bernstein polynomials can be generated by a recursive formula over an interval $[0,R]$, so that the ith Nth degree Bernstein polynomial can be written as

$$B_{i,N}(x)=\frac{R-x}{R}B_{i,N-1}(x)+\frac{x}{R}B_{i-1,N-1}(x)$$

As also it is mentioned in [3], any polynomial of degree n can be expanded in terms of a linear combination of the basis functions:

$$P(x)=\sum_{N}^{i=0} C_iB_{i,N}(x), \quad N\ge 1.$$

3 Fundamental Matrix Relations

We can convert the Bernstein series solutions $y_i(x)$ given with (3) and their derivatives $y_i^{(r)}(x)$ in to matrix forms

$$y_i(x)=B_N(x)A_i, \quad i=1,2,\cdots,k \tag{4}$$

and

$$y_i^{(r)}(x)=B_N^{(r)}(x)A_i \quad i=1,2,\cdots,k \tag{5}$$

where

$$B_N(x)=\begin{bmatrix}B_{0,N}(x) & B_{1,N}(x) & \cdots & B_{N,N}(x)\end{bmatrix}, A_i=\begin{bmatrix}y_{0,i} & y_{1,i} & \cdots & y_{N,i}\end{bmatrix}^T.$$

$B_N(x)$ can be represented as

$$B_N(x)=X(x).B \tag{6}$$

where

$$X(x)=\begin{bmatrix}1 & x & x^2 & \cdots & x^N\end{bmatrix}^T, \quad B=\begin{bmatrix}b_{00} & b_{01} & \cdots & b_{0N}\\ b_{10} & b_{11} & \cdots & b_{1N}\\ \vdots & \vdots & \ddots & \vdots\\ b_{N0} & b_{N1} & \cdots & b_{NN}\end{bmatrix} \text{ and } b_{ij}=\begin{cases}\frac{(-1)^{i-j}}{R^i}\binom{N}{j}\binom{N-j}{i-j}, & i\ge j\\ 0, & i<j\end{cases}.$$

For the matrix representations of the derivatives of approximate solutions we need the relation between $X(x)$ and its derivative $X^{(1)}(x)$ which can be clearly seen as

$$X^{(1)}(x)=X(x)D \tag{7}$$

where

$$D=\begin{bmatrix}0&1&0&0&\cdots&0\\0&0&2&0&\cdots&0\\0&0&0&3&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\cdots&N\\0&0&0&0&\cdots&0\end{bmatrix}$$

We obtain all the derivatives $X^{(r)}(x)$ in terms of $X(x)$ with the help of (7) by following calculations below:

$$\begin{gathered} X^{(2)}(x) = X^{(1)}(x)D = X(x)D^2 \\ \vdots \\ X^{(r)}(x) = X^{(r-1)}(x)D = X(x)D^r \end{gathered} \tag{8}$$

We reach similar relations between $X(\lambda x+\beta)$ and its derivatives $X^{(r)}(\lambda x+\beta)$ by using (8) given as follows:

$$X(\lambda x+\beta) = [1 \quad \lambda x+\beta \quad (\lambda x+\beta)^2 \cdots (\lambda x+\beta)^N] = X(x)C(\lambda,\beta) \tag{9}$$

$$\begin{gathered} X^{(1)}(\lambda x+\beta) = X(\lambda x+\beta)D = X(x)C(\lambda,\beta)D \\ X^{(2)}(\lambda x+\beta) = X(\lambda x+\beta)D^2 = X(x)C(\lambda,\beta)D^2 \\ \vdots \\ X^{(r)}(\lambda x+\beta) = X(\lambda x+\beta)D^r = X(x)C(\lambda,\beta)D^r \end{gathered} \tag{10}$$

where for $\lambda \neq 0$ and $\beta \neq 0$ $C(\lambda,\beta)$ is defined as

$$C(\lambda,\beta) = \begin{bmatrix} \binom{0}{0}\lambda^0\beta^0 & \binom{1}{0}\lambda^0\beta^1 & \cdots & \binom{N}{0}\lambda^0\beta^N \\ 0 & \binom{1}{1}\lambda^1\beta^0 & \cdots & \binom{N}{1}\lambda^1\beta^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}\lambda^N\beta^0 \end{bmatrix}$$

and for the values $\lambda = 1$ and $\beta = 0$ defined as

$$C(\lambda,0) = \begin{bmatrix} \lambda^0 & 0 & \cdots & 0 \\ 0 & \lambda^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^N \end{bmatrix}$$

Substituting (6) into (4) we get the matrix representation of y_i in (4) as

$$y_i(x) = X(x)BA_i, \quad i = 1,2,\cdots,k \tag{11}$$

Differentiating (11) consecutively with respect to x and using (8) we get the matrix expression of $y_i^{(r)}$ in (5) as follows:

$$y_i^{(r)}(x) = X(x)D^rBA_i, \quad i = 1,2,\cdots,k \tag{12}$$

By means of (9), (10), (11) and (12) we obtain the matrix relations

$$y_i^{(r)}(\lambda x+\beta) = X(x)C(\lambda,\beta)D^rBA_i, \quad i = 1,2,\cdots,k \tag{13}$$

Let us define the matrices

$$y^{(r)}(\lambda x+\beta) = \begin{bmatrix} y_1^{(r)}(\lambda x+\beta) \\ y_2^{(r)}(\lambda x+\beta) \\ \vdots \\ y_k^{(r)}(\lambda x+\beta) \end{bmatrix} \qquad r = 0,1,\cdots,m \tag{14}$$

which can be expressed by means of the relations given with (12) as

$$y^{(r)}(\lambda x+\beta) = X^*(x)C^*(\lambda,\beta)\widetilde{D}^r\widetilde{B}A, \quad r = 0,1,\cdots,m \tag{15}$$

where

$$X^*(x)=\begin{bmatrix} X(x) & 0 & \cdots & 0 \\ 0 & X(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X(x) \end{bmatrix},\ \widetilde{D}^r=\begin{bmatrix} D^r & 0 & \cdots & 0 \\ 0 & D^r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^r \end{bmatrix},$$

$$\widetilde{B}=\begin{bmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B \end{bmatrix},\ C^*(\lambda,\beta)=\begin{bmatrix} C(\lambda,\beta) & 0 & \cdots & 0 \\ 0 & C(\lambda,\beta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C(\lambda,\beta) \end{bmatrix} \text{ and } A=\begin{bmatrix} A_1 & A_2 & \cdots & A_k \end{bmatrix}^T.$$

4 Method of Solution

We can express the system given with (1) in the matrix form

$$\sum_{r=0}^{m} P_r(x)y^{(r)}(\lambda x+\beta)=f(x) \tag{16}$$

where $y^{(r)}(\lambda x+\beta)$ is as the form (14), $f(x)=\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_k(x)\end{bmatrix}^T$ and $P_r(x)=\begin{bmatrix} P^r_{1,1}(x) & P^r_{1,2}(x) & \cdots & P^r_{1,k}(x) \\ P^r_{2,1}(x) & P^r_{2,2}(x) & \cdots & P^r_{2,k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ P^r_{k,1}(x) & P^r_{k,2}(x) & \cdots & P^r_{k,k}(x) \end{bmatrix}$. By substituting the node points $\{x_i|0=x_0<x_1<\cdots<x_N=R,\ i=0,1,\cdots,N\}$ into the matrix equation (16) we obtain the system of fundamental matrix equation as

$$\sum_{r=0}^{m} P_r(x)Y^{(r)}=F \tag{17}$$

where $P_r=\begin{bmatrix} P_r(x_0) & 0 & \cdots & 0 \\ 0 & P_r(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_r(x_N) \end{bmatrix}$, $\quad Y^{(r)}=\begin{bmatrix} y^{(r)}(\lambda x_0+\beta) \\ y^{(r)}(\lambda x_1+\beta) \\ \vdots \\ y^{(r)}(\lambda x_N+\beta) \end{bmatrix}^T$ and $f(x)=\begin{bmatrix} f(x_0) & f(x_1) & \cdots & f(x_N)\end{bmatrix}^T$.
Using the node points and (15) we can rewrite $Y^{(r)}$ in the matrix form

$$Y^{(r)}=XC^*(\lambda,\beta)\widetilde{D}^r\widetilde{B}A, \quad r=0,1,\cdots,m \tag{18}$$

where

$$X=\begin{bmatrix} X^*(x_0) & X^*(x_1) & \cdots & X^*(x_N)\end{bmatrix}^T.$$

Substituting (18) into expression (17) we have the fundamental matrix equation

$$\left\{\sum_{r=0}^{m} P_rXC^*(\lambda,\beta)\widetilde{D}^r\widetilde{B}\right\}A=F \tag{19}$$

The fundamental matrix relation (19) corresponding to equation system (1) can be expressed in the following form

$$WA=F \text{ or } [W;F] \tag{20}$$

where

$$W=[W_{p,q}]=\sum_{r=0}^{m} P_rXC^*(\lambda,\beta)\widetilde{D}^r\widetilde{B}, \quad p,q=1,2,\cdots,k(N+1) \tag{21}$$

which is a linear system of $k(N+1)$ algebraic equations in $k(N+1)$ unknown Bernstein coefficients $a_{n,i}$, $n=0,1,\cdots,N$, $i=1,2,\cdots,k$. We can obtain the matrix representations of the conditions given with (2) by following a similar way used for the system (1). Writing the conditions in (2) for each n we have the following equations

$$\sum_{j=0}^{m-1} a^1_{r,j}y_1^{(j)}(a)+b^1_{r,j}y_1^{(j)}(b)+c^1_{r,j}y_1^{(j)}(c)=\lambda_{1,r}$$
$$\sum_{j=0}^{m-1} a^2_{r,j}y_2^{(j)}(a)+b^2_{r,j}y_2^{(j)}(b)+c^2_{r,j}y_2^{(j)}(c)=\lambda_{2,r}$$
$$\vdots$$
$$\sum_{j=0}^{m-1} a^k_{r,j}y_k^{(j)}(a)+b^k_{r,j}y_k^{(j)}(b)+c^k_{r,j}y_k^{(j)}(c)=\lambda_{k,r}$$

which we can also express as

$$\sum_{j=0}^{m-1} A_{0,j}y^{(j)}(a)+B_{0,j}y^{(j)}(b)+C_{0,j}y^{(j)}(c)=\lambda_0$$
$$\sum_{j=0}^{m-1} A_{1,j}y^{(j)}(a)+B_{1,j}y^{(j)}(b)+C_{1,j}y^{(j)}(c)=\lambda_1$$
$$\vdots$$
$$\sum_{j=0}^{m-1} A_{m-1,j}y^{(j)}(a)+B_{m-1,j}y^{(j)}(b)+C_{m-1,j}y^{(j)}(c)=\lambda_{m-1}$$

where

$$A_{r,j}=\begin{bmatrix} a^1_{r,j} & 0 & \cdots & 0 \\ 0 & a^2_{r,j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^k_{r,j} \end{bmatrix}, B_{r,j}=\begin{bmatrix} b^1_{r,j} & 0 & \cdots & 0 \\ 0 & b^2_{r,j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^k_{r,j} \end{bmatrix},$$
$$C_{r,j}=\begin{bmatrix} c^1_{r,j} & 0 & \cdots & 0 \\ 0 & c^2_{r,j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c^k_{r,j} \end{bmatrix} \text{ and } \lambda_r=\begin{bmatrix} \lambda_{1,r} & \lambda_{2,r} & \cdots & \lambda_{k,r} \end{bmatrix}^T,$$

for $r=0,1,\cdots,m-1$ or briefly

$$\sum_{j=0}^{m-1} A_jy^{(j)}(a)+B_jy^{(j)}(b)+C_jy^{(j)}(c)=\lambda \tag{22}$$

where

$$A_j=\begin{bmatrix} A_{0,j} & A_{1,j} & \cdots & A_{m-1,j} \end{bmatrix}^T, B_j=\begin{bmatrix} B_{0,j} & B_{1,j} & \cdots & B_{m-1,j} \end{bmatrix}^T,$$
$$C_j=\begin{bmatrix} C_{0,j} & C_{1,j} & \cdots & C_{m-1,j} \end{bmatrix}^T \text{ and } \lambda=\begin{bmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_{m-1} \end{bmatrix}^T.$$

By substituting $\lambda=1$ and $\beta=0$ in (14) and calculating this matrix at points a, b and c, we get the matrix representations of $y^j(a)$, $y^j(b)$ and $y^j(c)$ in (22) as follows:

$$y^{(j)}(a)=X^*(a)\widetilde{D}^jA$$
$$y^{(j)}(b)=X^*(b)\widetilde{D}^jA$$
$$y^{(j)}(c)=X^*(c)\widetilde{D}^jA \tag{23}$$

Substituting the matrices given in (23) in the equation (22) and simplifying the result we obtain the matrix representation of the conditions containing the coefficient matrix A as

$$\lambda = \sum_{j=0}^{m-1} [A_j X^*(a) + B_j X^*(b) + C_j X^*(c)]\widetilde{D}^j A \tag{24}$$

Defining the matrix

$$V = \sum_{j=0}^{m-1} [A_j X^*(a) + B_j X^*(b) + C_j X^*(c)]\widetilde{D}^j \tag{25}$$

we can write the matrix form of the conditions given with (24) as

$$VA = \lambda \tag{26}$$

Finally by replacing number of mk rows of the matrix W and F with the rows of the matrix V and λ, respectively, we obtain the unknown coefficients of the approximate solutions of the system (1). Generally for simplicity we prefer to use the last rows of the matrices for replacement and in this we illustrate this replacement procedure as follows:

$$\widetilde{W}A = \widetilde{F} \tag{27}$$

where

$$\widetilde{W} = \begin{bmatrix} W_{1,1} & W_{1,2} & \cdots & W_{1,k(N+1)} \\ W_{2,1} & W_{2,2} & \cdots & W_{2,k(N+1)} \\ \vdots & \vdots & \vdots & \vdots \\ W_{k(N-m+1),1} & W_{k(N-m+1),2} & \cdots & W_{k(N-m+1),k(N+1)} \\ V_{1,1} & V_{1,2} & \cdots & V_{1,k(N+1)} \\ V_{2,1} & V_{2,2} & \cdots & V_{2,k(N+1)} \\ \vdots & \vdots & \vdots & \vdots \\ V_{km,1} & V_{km,2} & \cdots & V_{km,k(N+1)} \end{bmatrix}$$

and $\widetilde{F} = \left[f_1(x_0) \cdots f_k(x_0) \; f_1(x_1) \cdots f_k(x_{N-m}) \; \lambda_{1,0} \cdots \lambda_{1,m-1} \; \lambda_{2,0} \cdots \lambda_{k,m-1} \right]^T$. If the matrix W is singular, then the rows having the same factor or all zero are replaced. Hence A is obtained as

$$A = (\widetilde{W})^{-1}\widetilde{F}. \tag{28}$$

5 Error Estimation Based on Residual Function

Researchers used the residual error estimation for residual correction [5,11], the error estimation of the Tau method for integro-differential equations [14], error estimation of the Bessel collacation method for the multi-pantograph equations [18], Laguerre matrix method for delay differential equations [19]. In this section we present an error estimation based on residual function by modifying the error estimation studied in [5,11,14,18,19]. Let $e_{i,N}(x) = y_i(x) - y_{i,N}(x)$, $(i = 1, \cdots, k)$ denote the error function of approximate solution $y_{i,N}$ to y_i, where y_i is one of the exact solutions of the problem of equation system given with (1) and (2). Hence $y_{i,N}(x)$ satisfies the following problem which can be obtained by substituting the approximate solution $y_{i,N}$ into the problem given with (1) :

$$\sum_{r=0}^{m}\sum_{i=1}^{k} P_{j,i}^{r}(x) y_{i,N}^{(r)}(\lambda x + \beta) = f_j(x) + R_{j,N}(x), \quad x \in [a,b], \quad j = 1,2,\cdots,k \tag{29}$$

with the mixed conditions

$$\sum_{j=1}^{m-1} a_{r,j}^{n}(x) y_{n,N}^{(j)}(a) + b_{r,j}^{n}(x) y_{n,N}^{(j)}(b) + c_{r,j}^{n}(x) y_{n,N}^{(j)}(c) = \lambda_{n,r}, \tag{30}$$
$$a \le c \le b, \quad r = 0,1,2,\cdots,m-1, \quad n = 1,2,\cdots,k$$

where $R_{j,N}$, $(n=1,2,\cdots,k)$ denote the residual functions associated with the approximate solutions $y_{j,N}$. Subtracting (29) and (30) from (1) and (2),respectively, the following error problem

$$\sum_{r=0}^{m}\sum_{i=1}^{k} P_{j,i}^{r}(x)e_{i,N}^{(r)}(\lambda x+\beta) = -R_{j,N}(x), \quad x\in[a,b],\ j=1,2,\cdots,k \tag{31}$$

with the mixed conditions

$$\sum_{j=1}^{m-1} a_{r,j}^{n}(x)e_{n,N}^{(j)}(a)+b_{r,j}^{n}(x)e_{n,N}^{(j)}(b)+c_{r,j}^{n}(x)e_{n,N}^{(j)}(c)=0, \tag{32}$$
$$a\le c\le b, \quad r=0,1,2,\cdots,m-1, \quad n=1,2,\cdots,k$$

is obtained which is satisfied by the error functions $e_{i,N}$, $(i=1,2,\cdots,k)$. Solving this error problem with the same method given in section 4 the approximation $e_{i,N,M}$ to error $e_{i,N}$ is attained. The advantage of solving this error problem is that the error of the approximate solution can be estimated without knowing the exact solution. In the next section we prefer to use examples whose exact solutions are known in order to compare the exact errors with the estimated errors.

6 Numerical Examples

*Example 1.*Consider the following linear differential-difference equation system and the conditions given by

$$\begin{gathered}
-xy_1^{(1)}(x+1)+2y_1(x+1)+2y_2(x+1)=x+7\\
2y_2(x+1)-3y_1^{(1)}(x+1)-4y_1^{(2)}(x+1)=0\\
y_1(1/2)=3/4, y_2(1/2)=11/2, y_1(1)=0, y_2(1)=7
\end{gathered}$$

where $0\le x\le 1$. For $N=3$ the nodes are $x_0=0, x_1=\frac{1}{3}, x_2=\frac{2}{3}, x_4=1$. Fundamental matrix equation of the problem is

$$\{P_0XC^*(1,1)\widetilde{B}+P_1XC^*(1,1)\widetilde{D}\widetilde{B}+P_2XC^*(1,1)\widetilde{D}^2\widetilde{B}\}A=F$$

where

$$P_0=\begin{bmatrix}2&1&0&0&0&0&0&0\\0&2&0&0&0&0&0&0\\0&0&2&1&0&0&0&0\\0&0&0&2&0&0&0&0\\0&0&0&0&2&1&0&0\\0&0&0&0&0&2&0&0\\0&0&0&0&0&0&2&1\\0&0&0&0&0&0&0&2\end{bmatrix}, P_1=\begin{bmatrix}0&0&0&0&0&0&0&0\\-3&0&0&0&0&0&0&0\\0&0&-\frac{1}{3}&0&0&0&0&0\\0&0&-3&0&0&0&0&0\\0&0&0&0&-\frac{2}{3}&0&0&0\\0&0&0&0&-3&0&0&0\\0&0&0&0&0&0&-1&0\\0&0&0&0&0&0&-3&0\end{bmatrix},$$

$$P_2=\begin{bmatrix}0&0&0&0&0&0&0&0\\-4&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&-4&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&-4&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-4&0\end{bmatrix}, X=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\1&\frac{1}{3}&\frac{1}{9}&\frac{1}{27}&0&0&0&0\\0&0&0&0&1&\frac{1}{3}&\frac{1}{9}&\frac{1}{27}\\1&\frac{2}{3}&\frac{4}{9}&\frac{8}{27}&0&0&0&0\\0&0&0&0&1&\frac{2}{3}&\frac{4}{9}&\frac{8}{27}\\1&1&1&1&0&0&0&0\\0&0&0&0&1&1&1&1\end{bmatrix},$$

$$\widetilde{B}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -6 & 3 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -3 & 1 \end{bmatrix}, C^{*}(1,1)=\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\widetilde{D}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, F=\begin{bmatrix} 7 \\ 0 \\ \frac{22}{3} \\ 0 \\ \frac{23}{3} \\ 0 \\ \frac{23}{3} \\ 0 \end{bmatrix}, A=\begin{bmatrix} A_1 \\ A_2 \end{bmatrix},$$

$$A_1=\begin{bmatrix} y_{0,1} \\ y_{1,1} \\ y_{2,1} \\ y_{3,1} \end{bmatrix}, A_2=\begin{bmatrix} y_{0,2} \\ y_{1,2} \\ y_{2,2} \\ y_{3,2} \end{bmatrix}.$$

Using the above matrices we obtain the matrix W defined with (21) as

$$W=\begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & -24 & 57 & -33 & 0 & 0 & 0 & 2 \\ \frac{1}{27} & \frac{-1}{9} & \frac{-8}{9} & \frac{80}{27} & \frac{-1}{27} & \frac{4}{9} & \frac{-16}{9} & \frac{64}{27} \\ 9 & -57 & 96 & -48 & -\frac{2}{27} & \frac{8}{9} & -\frac{32}{9} & \frac{128}{27} \\ \frac{8}{27} & -\frac{8}{9} & -\frac{10}{9} & \frac{100}{27} & -\frac{8}{27} & \frac{20}{9} & -\frac{50}{9} & \frac{125}{27} \\ 20 & -96 & 141 & 65 & -\frac{16}{27} & \frac{40}{9} & -\frac{100}{9} & \frac{250}{27} \\ 1 & -3 & 0 & 1 & 4 & 6 & -12 & 8 \\ 33 & -141 & 192 & -84 & -2 & 12 & -24 & 16 \end{bmatrix}.$$

From (24), (25) and (26) matrix representation of the given conditions are obtained as

$$V=\begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & 0 & -3 & 3 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 \end{bmatrix}, \lambda=\begin{bmatrix} -\frac{3}{4} \\ \frac{11}{2} \\ 0 \\ 7 \end{bmatrix}, VA=\lambda.$$

By performing the row replacement we obtain the matrices $\widetilde{W}$ and $\widetilde{F}$ in (27) as follows:

$$\widetilde{W}=\begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & -24 & 57 & -33 & 0 & 0 & 0 & 2 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & 0 & -3 & 3 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 \end{bmatrix}, \widetilde{F}=\begin{bmatrix} 7 \\ 0 \\ \frac{22}{3} \\ -\frac{3}{4} \\ \frac{11}{2} \\ 0 \\ 7 \end{bmatrix}.$$

By using (28), the unknown coefficient matrix A is obtained as

$$A=\begin{bmatrix} -1 & -1 & -\frac{2}{3} & 0 & 4 & 5 & 6 & 7 \end{bmatrix}^{T}.$$

Hence substituting appropriate coefficients in (4) we obtain the approximate solutions as

$$y_1(x)=x^2-1, y_2(x)=3x+1.$$

which are the same as exact solutions.

Example 2.([8], [17], [20]) Consider the following linear differential system and the condition given with:

$$y_1^{(1)}(x)+y_2^{(1)}(x)+y_1(x)+y_2(x)=1$$
$$y_2^{(1)}(x)-2y_1(x)-y_2(x)=0$$
$$y_1(0)=0, y_2(0)=1$$

where $0\leq x\leq 1$. Exact solutions of this system are $y_1(x)=e^{-x}-1$ and $y_2(x)=2-e^{-x}$. For this example we can align the basic values of the problem as $k=2, m=1, \lambda=1, \beta=0, f_1(x)=1, f_2(x)=0$, $P_{1,1}^0=1$, $P_{1,2}^0=1$, $P_{2,1}^0=-2, P_{2,2}^0=-1, P_{1,1}^1=1, P_{1,2}^1=1, P_{2,1}^1=0, P_{2,2}^1=1$. Fundamental matrix equation of the problem is

$$\{P_0XC^*(1,0)\widetilde{B}+P_1XC^*(1,0)\widetilde{D}\widetilde{B}\}A=F$$

Following the method given in section 4 we obtain the approximate solutions for $i=1,2$ and $N=6,8,10$ as

$$y_{1,6}(x)=-t+0.49999287763359t^2-0.1665992415988t^3+(0.414094305449e-1)t^4 -(0.78459973259e-2)t^5+(0.92305849257e-3)t^6 ,$$

$$y_{2,6}(x)=1+1.00000000000002t-0.4999928776345t^2+0.1665992416052t^3 -(0.414094305609e-1)t^4+(0.78459973420e-2)t^5-(0.92305849852e-3)t^6,$$

$$y_{1,8}(x)=-t+0.49999998063417t^2-0.1666663924244t^3+(0.416649836193e-1)t^4 -(0.8327671049e-2)t^5+(0.13776655847e-2)t^6-(0.1852302697e-3)t^7 +(0.16106411517e-4)t^8 ,$$

$$y_{2,8}(x)=1+t-0.4999999806306t^2+0.1666663923794t^3-(0.416649833725e-1)t^4 +(0.8327670334e-2)t^5-(0.13776644215e-2)t^6+(0.1852292920e-3)t^7 -(0.1610607624e-4)t^8 ,$$

$$y_{1,10}(x)=-t+0.49999999996797221464t^2-0.16666666605195217 2t^3 +(0.41666661337852444e-1)t^4-(0.833330671871991e-2)t^5+(0.138880521754404e-2)t^6 -(0.1982411886115e-3)t^7+(0.245716339029e-4)t^8-(0.2559545198236e-5)t^9 +(0.17652035850008e-6)t^{10} ,$$

$$y_{2,10}(x)=1+t-0.499999999967972145t^2+0.16666666605195216t^3 -(0.4166666133785242e-1)t^4+(0.83333067187197e-2)t^5-(0.13888052175437e-2)t^6 +(0.1982411886110e-3)t^7-(0.24571633902642e-4)t^8+(0.25595451979e-5)t^9 -(0.1765203584601e-6)t^{10} .$$

Comparison of absolute error values with the ones in [8], [17] and [20] are given in tables 1 and 3. As seen from the tables we obtain better results rather than Transform method [17] and we obtain very similar results to Bessel Method [20] and Taylor method [8] results. The estimated errors for the error functions $e_{1,6}$ and $e_{2,6}$ are given in tables 2 and 4 from which we can see that our estimations are very close to exact errors. In Figure 1 graphs of the error functions $e_{1,N}$ and $e_{2,N}$ for the values $N=6,8,10$ are given. It is clearly seen from these graphs that errors are decreasing as the value of N increases.

Table 1: Numerical results of absolute error function $e_{1,6}$ of Example 2

x_i	Transform Method [17]	Bessel Method [20]	Taylor Method [8]	Present Method
0	0	0	0	0
0.2	1.5575e-002	2.8460e-008	2.845945e-008	2.845996e-008
0.4	5.1209e-002	1.7820e-008	1.782165e-008	1.781967e-008
0.6	1.0150e-001	1.2668e-008	1.267680e-008	1.266769e-008
0.8	1.6630e-001	3.3538e-008	3.356743e-008	3.353739e-008
1	2.3351e-001	6.8657e-007	6.865716e-007	6.86574918e-007

Table 2: Comparison of numerical results of absolute error function $e_{1,6}$ and estimated error functions $e_{1,N,M}$ for $M = 8, 10, 12$ of Example 2

x_i	$e_{1,6}$	$e_{1,6,8}$	$e_{1,6,10}$	$e_{1,6,12}$
0	0	0	0	0
0.2	2.845996e-008	2.84279073056582e-008	2.84599272455068e-008	2.84599575591665e-008
0.4	1.781967e-008	1.77919244485051e-008	1.78196446634016e-008	1.78196702909098e-008
0.6	1.266769e-008	1.2643882723471e-008	1.26676677388808e-008	1.26676884155990e-008
0.8	3.353739e-008	3.351717499016e-008	3.35373810636526e-008	3.35373931840587e-008
1	6.86574918e-007	6.852371508769e-007	6.86573211575897e-007	6.86574916204103e-007

Table 3: Numerical results of absolute error function $e_{2,6}$ of Example 2

x_i	Transform Method [17]	Bessel Method [20]	Taylor Method [8]	Present Method
0	0	0	0	0
0.2	2.3262e-003	2.8460e-008	2.84595e-008	2.8459957e-008
0.4	1.6867e-002	1.7820e-008	1.782165e-008	1.7819678e-008
0.6	5.4499e-002	1.2668e-008	1.26768e-008	1.2667692e-008
0.8	1.3194e-001	3.3538e-008	3.35675e-008	3.3537418e-008
1	2.8038e-001	6.8657e-007	6.865717e-007	6.86574798e-007

Table 4: Comparison of numerical results of absolute error function $e_{2,6}$ and estimated error functions $e_{1,N,M}$ for $M = 8, 10, 12$ of Example 2

x_i	$e_{2,6}$	$e_{2,6,8}$	$e_{2,6,10}$	$e_{2,6,12}$
0	0	0	0	0
0.2	2.8459957e-008	2.84279054342991e-008	2.84599253741479e-008	2.84599556878058e-008
0.4	1.7819678e-008	1.77919322974740e-008	1.78196525123913e-008	1.78196781398054e-008
0.6	1.2667692e-008	1.2643887269948e-008	1.26676722855722e-008	1.26676929615788e-008
0.8	3.3537418e-008	3.351720103513e-008	3.35374071096589e-008	3.35374192277675e-008
1	6.86574798e-007	6.852370308781e-007	6.86573091574497e-007	6.86574796213403e-007

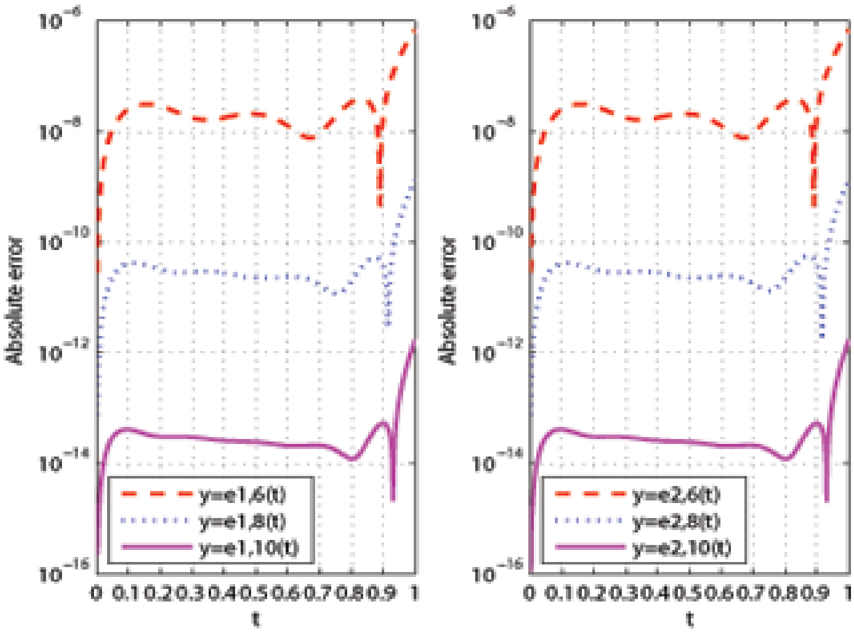

Fig. 1: Graphs of absolute error functions $e_{1,N}$ and $e_{2,N}$ for values $N = 6,8,10$ of Example 2

Example 3.([2], [6], [8], [20]) Consider the following linear differential system and the condition given with:

$$y_1^{(1)}(x)+y_2^{(1)}(x)+y_2(x)=x-e^{-x}$$
$$y_1^{(1)}(x)+4y_2^{(1)}(x)+y_1(x)=1+2e^{-x}$$
$$y_1(0)=1, y_2(0)=0$$

where $0 \le x \le 1$. Exact solutions of this system are $y_1(x)=e^{-x}+3e^{-x/3}-3$ and $y_2(x)=-(1/2)e^{-x}+(3/2)e^{-x/3}-1+x$. Fundamental matrix equation of the problem is

$$\{P_0XC^*(1,0)\widetilde{B}+P_1XC^*(1,0)\widetilde{D}\widetilde{B}\}A=F$$

Following the method given in section 4 we obtain the approximate solutions for $i=1,2$ and $N=5,7,10$ as

$$y_{1,5}(x)=1-2t+0.6665429600045t^2-0.1843402224992t^3+(0.409518052792e-1)t^4 \\ -(0.569482927692e-2)t^5 \quad ,$$

$$y_{2,5}(x)=t-0.16660539022198t^2+(0.736556629852e-1)t^3-(0.189435481094e-1)t^4 \\ +(0.275727613671e-2)t^5 \quad ,$$

$$y_{1,7}(x)=1-2t+0.6666662244872t^2-0.18518020404123t^3+(0.431857423289e-1)t^4 \\ -(0.837418910669e-2)t^5+(0.13056961684e-2)t^6-(0.129930152198e-3)t^7,$$

$$y_{2,7}(x)=t-0.16666644581866t^2+(0.740715862927e-1)t^3-(0.200496748775e-1)t^4 \\ +(0.40842486361e-2)t^5-(0.64718137131e-3)t^6+(0.64728997204e-4)t^7,$$

$$y_{1,10}(x) = 1 - 2t + 0.666666666631961166t^2 - 0.185185184538034076t^3 + (0.432098710082210 91e-1)t^4 - (0.843618659713503e-2)t^5 + (0.139451895946467e-2)t^6 - (0.1985105812431389e-3)t^7 + (0.24580409370444e-4)t^8 - (0.255863683414e-5)t^9 + (0.17623855808795e-6)t^{10},$$

$$y_{2,10}(x) = t - 0.166666666649314 6472t^2 + (0.74074073750512379e-1)t^3 - (0.20061725627686513e-1)t^4 + (0.411521264072175e-2)t^5 - (0.69154388943591e-3)t^6 + (0.989831232911e-4)t^7 - (0.1227886913003e-4)t^8 + (0.1278902468183e-5)t^9 - (0.8810722289409e-7)t^{10}$$

Comparison of absolute error values with the ones in [2], [6] and [20] are given in tables 5 and 7. We see from these tables that results obtained with present method are better rather than Chebyshev method [2], Stehfest method [6] and they are very close with Bessel Method [20] results. The estimated errors for the error functions $e_{1,6}$ and $e_{2,6}$ are given in tables 6 and 8 from which we can see that our estimations are very close to exact errors. In Figure 2 graphs of the error functions $e_{1,N}$ and $e_{2,N}$ for the values $N = 5,7,10$ are given. It is clearly seen from these graphs that errors are decreasing as the value of N increases.

Table 5: Numerical results of absolute error function $e_{1,5}$ of Example 3

x_i	Chebyshev Method [2]	Stehfest Method [6]	Bessel Method [20]	Present Method
0.1	4.510522e-005	6.7614e-005	5.9187e-007	5.9187220e-007
0.2	7.985043e-005	8.4949e-005	1.0110e-006	1.01100956e-006
0.5	9.719089e-005	3.18972e-003	1.0978e-006	1.09778070e-006
0.8	8.006002e-005	5.20283e-003	9.0094e-007	9.0094297e-007
1	1.067677e-004	1.193776e-002	1.3659e-005	1.365938523e-005

Table 6: Comparison of numerical results of absolute error function $e_{1,5}$ and estimated error functions $e_{1,N,M}$ for $M = 8,10,12$ of Example 3

x_i	$e_{1,5}$	$e_{1,5,8}$	$e_{1,5,10}$	$e_{1,5,12}$
0.1	5.9187220e-007	5.91822648308304e-007	5.91872145817983e-007	5.91872196460526e-007
0.2	1.01100956e-006	1.01096027100425e-006	1.01100952299922e-006	1.01100957076403e-006
0.5	1.09778070e-006	1.09771587207109e-006	1.09778063704070e-006	1.09778070501075e-006
0.8	9.0094297e-007	9.0085173517667e-007	9.00942894065060e-007	9.00942980477876e-007
1	1.365938523e-005	1.36605388297924e-005	1.36593867114888e-005	1.36593852717651e-005

Table 7: Numerical results of absolute error function $e_{2,5}$ of Example 3

x_i	Chebyshev Method [2]	Stehfest Method [6]	Bessel Method [20]	Present Method
0.1	2.247723e-005	8.4086e-006	2.9327e-007	2.9327369e-007
0.2	3.984701e-005	1.9575e-005	5.0134e-007	5.0133795e-007
0.5	4.890662e-005	2.242e-004	5.4596e-007	5.4596049e-007
0.8	4.064222e-005	4.647e-004	4.5116e-007	4.5115686e-007
1	5.390356e-005	4.710e-004	6.7555e-006	6.755515571e-006

Table 8: Comparison of numerical results of absolute error function $e_{2,5}$ and estimated error functions $e_{1,N,M}$ for $M = 8, 10, 12$ of Example 3

x_i	$e_{2,5}$	$e_{2,5,8}$	$e_{2,5,10}$	$e_{2,5,12}$
0.1	2.9327369e-007	2.93248921695193e-007	2.93273661159323e-007	2.93273687831747e-007
0.2	5.0133795e-007	5.01313313249189e-007	5.01337931456269e-007	5.01337956360630e-007
0.5	5.4596049e-007	5.45928075723713e-007	5.45960451215797e-007	5.45960486276556e-007
0.8	4.5115686e-007	4.5111124446184e-007	4.51156817203302e-007	4.51156861406332e-007
1	6.755515571e-006	6.75609209698840e-006	6.75551629128592e-006	6.75551564509039e-006

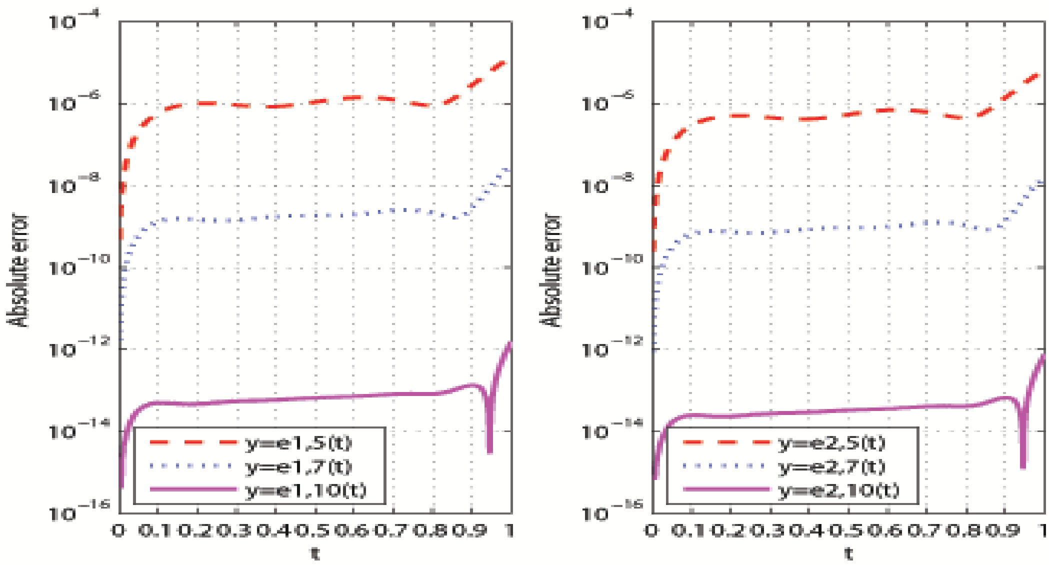

Fig. 2: Graphs of absolute error functions $e_{1,N}$ and $e_{2,N}$ for values $N = 5, 7, 10$ of Example 3

7 Conclusions

As many kind of problems in applied mathematics are difficult to solve analytically, systems of high-order linear differential-difference equations are so. Because of this issue numerical approaches to solutions of these problem kind are frequently preferred. We choose to use Bernstein polynomials for making an approach to solutions of systems of high-order linear differential-difference equations with variable coefficients under mixed conditions by using nodes which are also called collacation points and converting the problem and the conditions into matrix forms. In some studies because of using collacation points these kind of methods are called as collacation methods as in [8]. Comparing the results in tables 1, 3, 5 and 7, we see that present method brings better results rather than Transform method [17], Chebyshev method [2] and Stehfest method [6] and have very close results with Bessel Method [20] and Taylor method [8]. From the figures 1 and 2 we also see that as N increases the absolute error values are decreasing. All these shows that our method is efficient and effective. In addition to solving the problem (1) given with the mixed conditions (2), we used residual error function in order to estimate the absolute errors and as seen from the tables 2, 4, 6 and 8 error estimation are very close real absolute errors. This part of our study is important for the approximating the results of the problems whose exact solution are unknown. An advantage of the method is that obtaining approximate solutions easy and fast by using Maple 15 algebraic computer program. The method can be developed to solve other problem types such as systems of linear integral and integro differential-difference equations by making required modifications.

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