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A multivariate rational interpolation with no poles in $\mathbb{R}^m$

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Abstract: The aim of this paper is to construct a family of rational interpolants that have no poles in $\mathbb{R}^m$. This method is an extension of Floater and Hormanns method [1]. A priori error estimate for the method is given under some regularity conditions.

Keywords: Rational interpolant, Floater and Hormanns method, Error estimate.

1 Introduction

Given a function f defined on an $m-$dimensional box, we can approximate f by polynomial interpolation. If the set of approximating functions is extended to the set of all rational functions, namely functions of the form $\dfrac{p}{q}$, where p and q are any polynomials, it is hard to control the occurrence of poles and to specify accuracy of the approximate solution. Polynomial interpolation is thus a special case of rational interpolation. Hence, it can be expected that rational interpolation may give better results then multivariate polynomial interpolation.

In $1-$dimension, Berrut and Mittelmann [2] suggested that it might be possible to avoid poles by using rational functions of higher degree. They considered algorithms which fit rational functions whose numerator and denominator degrees can both be as high as any positive integer n. As observed in Berrut and Mittelmann [2], every such interpolant can be given in the barycentric form

$$r(x)=\frac{\sum\limits_{i=0}^{n}\dfrac{w_i}{x-x_i}f(x_i)}{\sum\limits_{i=0}^{n}\dfrac{w_i}{x-x_i}}$$

for some real values w_i. Thus it is enough for good approximation rates to find the weights $w_0,w_1,\ldots,w_n$ to specify the function r. There was another suggestion by Berrut [3], simply to take

$$w_i=(-1)^i \ , \ k=0,1,\ldots,n$$

giving

$$r(x)=\frac{\sum\limits_{i=0}^{n}\dfrac{(-1)^i f(x_i)}{x-x_i}}{\sum\limits_{i=0}^{n}\dfrac{(-1)^i}{x-x_i}}. \tag{1}$$

Berrut showed that (1) has no poles in $\mathbb{R}$. See also Berrut [4],[5].

Floater and Hormann [1] reported that there is a whole family of barycentric rational interpolants with arbitrarily high approximation orders, including the interpolant (1) as a special case. The construction is as follows. Choose any integer

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d with $0 \le d \le n$, and for each $i = 0,1,...,n-d$, let p_i denote the unique polynomial of degree at most d that interpolates f at the $d+1$ points $x_i, x_{i+1}, \ldots, x_{i+d}$. Then let

$$r(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)} \tag{2}$$

where

$$\lambda_i(x) = \frac{(-1)^i}{(x-x_i)\ldots(x-x_{i+d})}.$$

For each $d = 0,1,2,...,n$, none of (2) has any poles in $\mathbb{R}$. In addition, for fixed $d \ge 1$ the interpolant has approximation order $O(h^{d+1})$ as $h \to 0$, where

$$h := \max_{0 \le i \le n-1} (x_{i+1} - x_i) \tag{3}$$

as long as $f \in C^{d+2}[a,b]$.

Floater and Hormann [1] used the following construction to show (2) that has no poles in $\mathbb{R}$. On multiplying the numerator and denominator in (2) by the product

$$(-1)^{n-d}(x-x_0)\ldots(x-x_n),$$

we obtain

$$r(x) = \frac{\sum_{i=0}^{n-d} \mu_i(x) p_i(x)}{\sum_{i=0}^{n-d} \mu_i(x)} \tag{4}$$

where

$$\mu_i(x) = (-1)^{n-d}(x-x_0)\ldots(x-x_n)\lambda_i(x),$$

which we can also express in the form

$$\mu_i(x) = \prod_{j=0}^{i-1}(x-x_j) \prod_{k=i+d+1}^{n}(x_k - x). \tag{5}$$

As usual, an empty product in (5) has value 1. After than, Floater and Hormann [1] analyzed the convergence and the results are given in the following theorems.

Theorem 1. *For all d, $0 \le d \le n$, the rational function r in (4) has no poles in $\mathbb{R}$.*

Theorem 2. *Suppose $d \ge 1$ and $f \in C^{d+2}[a,b]$, and let r be the rational function in (2) and h be as in (3). If $n-d$ is odd then*

$$\|r-f\|_\infty \le h^{d+1}(b-a)\frac{\left\|f^{(d+2)}\right\|_\infty}{d+2}. \tag{6}$$

If $n-d$ is even then

$$\|r-f\|_\infty \le h^{d+1}\left((b-a)\frac{\left\|f^{(d+2)}\right\|_\infty}{d+2} + \frac{\left\|f^{(d+1)}\right\|_\infty}{d+1}\right). \tag{7}$$

Theorem 3. *Suppose $d = 0$, $f \in C^2[a,b]$ and the local mesh ratio*

$$\beta := \max_{1 \le i \le n-2} \min\left\{\frac{x_{i+1}-x_i}{x_i - x_{i-1}}, \frac{x_{i+1}-x_i}{x_{i+2}-x_{i+1}}\right\}$$

is bounded when $h \to 0$. Let r be the rational function in (2). If n is odd then

$$\|r-f\|_\infty \le h(1+\beta)(b-a)\frac{\|f''\|_\infty}{2}.$$

If n is even then

$$\|r-f\|_\infty \le h(1+\beta)\left((b-a)\frac{\|f''\|_\infty}{2}+\|f'\|_\infty\right).$$

In this study, as an extension of the methods of Floater and Hormann [1] in the univariate case, a family of multivariate rational interpolants which have no poles in $\mathbb{R}^m$ is constructed. The method is given with a priori error estimate under low regularity assumptions. We note that the set of interpolation nodes K form a tensor product grid.

2 Preliminaries

2.1 Interpolation Polynomials

Let us consider the $n+1$ pairs x_i in $\mathbb{R}$. The problem is to find an interpolating polynomial such that

$$p_m f(x_i) = a_0 + a_1 x_i + \cdots + a_m x_i^m = y_i\ ,\ i = 0,1,\ldots,n.$$

The points x_i are called interpolation nodes. The following two theorems can be found any numerical analysis text, (see, e.g., Quarteroni et al. [6]).

Theorem 4. *Given $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ and $n+1$ corresponding values $y_0, y_1, \ldots y_n$, then there exists a unique polynomial $p_n f \in P_n$ such that $p_n f(x_i) = y_i$ for $i = 0,1,\ldots,n$.*

In the next theorem, Lagrange characteristic polynomials are used which are defined as

$$l_i \in P_n\ :\ l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)},\quad i = 0,1,\ldots,n.$$

Theorem 5. *Let x and the abscissas $x_0, x_1, \ldots, x_n$ be contained in an interval $[a,b]$ on which f and its first $n+1$ derivatives are continuous. Then there exists $\xi_x \in (a,b)$, which depends on x, such that*

$$f(x) - p_n f(x) = \frac{1}{(n+1)!}\prod_{j=0}^{n}(x-x_j) f^{(n+1)}(\xi_x).$$

In $m-$dimensions, interpolation polynomial of a function f is defined similar to $1-$dimension, see e.g. [7]. Let us consider the n distinct points $\mathbf{x}_i = (x_i^1, \ldots, x_i^m)$ in $\mathbb{R}^m$. Let $\phi_1, \ldots, \phi_m$ denote m linearly independent functions in $C(\mathbb{R}^m)$. The interpolating problem is to determine $a_1, a_2, \ldots a_m$ such that

$$a_1\phi_1(\mathbf{x}_i) + a_2\phi_2(\mathbf{x}_i) + \cdots + a_m\phi_m(\mathbf{x}_i) = f(\mathbf{x}_i)$$

for $1 \le i \le n$.

The following paragraph was given by Mößner and Reif in [8].

The space dimension m is assumed as fixed and greater than 1. Let $K_j = \left\{x_j^i | 0 \le x_j^1 < x_j^2 < \cdots < x_j^{n_j} \le b_j, i = 1,\ldots,n_j\right\}$ denote a partition of the interval $[0,b_j]$. Given the tensor product partition $\mathbf{K} = \prod_{j=1}^{m} K_j$ of $\prod_{j=1}^{m}[0,b_j]$. For each coordinate direction $j = 1,2,\ldots,m$, univariate interpolation operator mapping a function f with an essentially bounded weak n_jth derivative to the unique polynomial $p_j = I_j f_j$ of order n_j, interpolating f_j on K_j, is denoted by

$$I_j : W_\infty^{n_j} \to P^{n_j} \tag{8}$$

Then, the interpolation polynomial can be written as

$$p_j(x) = \sum_{i=1}^{n_j} f(x_j^i) l_i^{n_j}(x),$$

where $l_1^{n_j}, \ldots, l_{n_j}^{n_j}$ are Lagrange polynomials. The error operator related to I_j is defined by $E_j := 1 - I_j$. Then, an upper bound for the error of polynomial interpolation can be written in the form

$$\left\| E_j f_j \right\|_j \leq w_j t_j^{n_j} \left\| f_j^{(n_j)} \right\|_j$$

where

$$w_j = \frac{\left\| (x - x_j^1) \cdots (x - x_j^{n_j}) \right\|_j}{t_j^{n_j} n_j!}$$

and $\|\|_j$ shows the supremum norm on $[0, b_j]$. For $m-$dimensions, let e_j denote the jth unit vector, and let $\mathbf{x}^j = \mathbf{x} - x_j e_j$. (8) is extended to an operator

$$I_j f(\mathbf{x}) = \sum_{i=1}^{n_j} f(\mathbf{x}^j + x_j^i e_j) l_i^{n_j}(\mathbf{x}) \ , \ \mathbf{x} \in \mathbf{K}$$

which acts only the jth component of a given multivariate function f and all other components are treated as constants. Then the tensor product interpolation

$$I := I_m \cdots I_1 : W_\infty^n(\mathbf{K}) \to P_n$$

interpolates f on $\mathbf{K}$ and $p := If$=$I_m \cdots I_1 f$ is unique. Let $\|\|$ denote sup-norm on $\mathbf{K}$. The error operator E can be given as

$$E = -\sum_{\|\alpha\|=1} (-E_m)^{\alpha_m} \cdots (-E_1)^{\alpha_1} \tag{9}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m$ is a multi-index with maximal component $\|\alpha\| = \max_i \alpha_i$. Thus, the upper bound of the error is obtained as

$$\|Ef\| \leq \sum_{\|\alpha\|=1} w_1^{\alpha_1} w_2^{\alpha_2} \cdots v_m^{\alpha_m} t_1^{\alpha_1 n_1} t_2^{\alpha_2 n_2} \ldots t_m^{\alpha_m n_m} \left\| \partial_1^{\alpha_1 n_1} \partial_2^{\alpha_2 n_2} \cdots \partial_m^{\alpha_m n_m} f \right\|_\infty$$

Let us write $\partial^\alpha f := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_m^{\alpha_m} f$ and $\mathbf{t}^\alpha := t_1^{n_1} t_2^{n_2} \ldots t_m^{n_m}$.

Theorem 6. *For $f \in W_\infty^n(\mathbf{K})$ the tensor product interpolation error on the box is bounded by*

$$\|f - If\| \leq \sum_{\|\alpha\|=1} \mathbf{w}^\alpha \mathbf{t}^{\alpha \mathbf{n}} \|\partial^{\alpha \mathbf{n}} f\|,$$

where $\mathbf{w} = (w_1, w_2, \ldots, w_m)$, $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ and $\alpha \mathbf{n} = (\alpha_1 n_1, \alpha_2 n_2, \ldots, \alpha_m n_m)$.

3 Rational interpolating function in $m-$dimensions

We will seek an approximate rational function for the given function f. Let $d_j \in \mathbb{Z}$ and $0 \leq d_j \leq n$. For $1 \leq j \leq m$ and $0 \leq i_j \leq n - d_j$, let $p_{i_1, i_2 \ldots, i_m}$ be the polynomial which interpolates f at

$$\left\{ (x_1^{t_1}, x_2^{t_2}, \ldots, x_m^{t_m}) : i_j \leq t_j \leq i_j + d_j, 1 \leq j \leq m \right\}.$$

Let

$$r = \frac{\sum_{i_1=0}^{n-d_1} \sum_{i_2=0}^{n-d_2} \cdots \sum_{i_m=0}^{n-d_m} \lambda_{i_1, i_2, \ldots, i_m} p_{i_1, i_2, \ldots, i_m}}{\sum_{i_1=0}^{n-d_1} \sum_{i_2=0}^{n-d_2} \cdots \sum_{i_m=0}^{n-d_m} \lambda_{i_1, i_2, \ldots, i_m}}, \tag{10}$$

where

$$\lambda_{i_1,i_2,\ldots,i_m}(\mathbf{x}) = \frac{(-1)^{\sum\limits_{k=1}^{m} i_k}}{\prod\limits_{i=i_1}^{i_1+d_1}\left(x_1 - x_1^j\right)\cdots\prod\limits_{i=i_m}^{i_m+d_m}\left(x_m - x_m^j\right)} \tag{11}$$

We will show that the function r in (10) is defined in $\mathbb{R}^m$. First, let us define a new function $\mu_{i_1,\ldots,i_m}$ as

$$\mu_{i_1,\ldots,i_m}(\mathbf{x}) = \prod_{k=0}^{i_1-1}\left(x_1 - x_1^k\right)\prod_{k=i_1+d_1+1}^{n}\left(x_1^k - x_1\right) \cdots \prod_{k=0}^{i_m-1}\left(x_m - x_m^k\right)\prod_{k=i_m+d_m+1}^{n}\left(x_m^k - x_m\right).$$

If the index set is empty, we will again understand its product has the value 1. On multiplying both numerator and denominator by

$$(-1)^{nm-\sum\limits_{k=0}^{m} d_k}\prod_{k=0}^{n}\left(x_1 - x_1^k\right)\cdots\prod_{k=0}^{m}\left(x_m - x_m^k\right)$$

we obtain

$$r = \frac{\sum\limits_{i_1=0}^{n-d_1}\sum\limits_{i_2=0}^{n-d_2}\cdots\sum\limits_{i_m=0}^{n-d_m}\mu_{i_1,i_2,\ldots,i_m}p_{i_1,i_2,\ldots,i_m}}{\sum\limits_{i_1=0}^{n-d_1}\sum\limits_{i_2=0}^{n-d_2}\cdots\sum\limits_{i_m=0}^{n-d_m}\mu_{i_1,i_2,\ldots,i_m}}. \tag{12}$$

The denominator of (12) is equal to

$$\sum_{i_1=0}^{n-d_1}\mu_i(x_1)\cdots\sum_{i_m=0}^{n-d_m}\mu_i(x_m).$$

We see from Theorem 1 that each component of the denominator is greater than zero, and the following results hold.

Theorem 7. *For all $d_1,d_2,\ldots,d_m$, $0 \le d_1,d_2,\ldots,d_m \le n$, the rational function r in (12) has no poles in $\prod\limits_{i=1}^{m}[a_i,b_i]$.*

Corollary 1. *For all $d_1,d_2,\ldots,d_m$, $0 \le d_1,d_2,\ldots,d_m \le n$, the rational function r in (12) has no poles in $\mathbb{R}^m$.*

Let us consider the rational interpolation function r. Let the interpolation polynomials in (10) are all be tensor product interpolants. First, let us discuss the convergence order for 2−dimensions.

Let $d_1,d_2 > 0$. Let I_1, I_2 and I_3 be defined as

$I_{1,k} = \{i : i \le \alpha - d_k, 0 \le i \le n - d_k\}$
$I_{2,k} = \{i : \alpha - d_k + 1 \le i \le \alpha, 0 \le i \le n - d_k\}$,
$I_{3,k} = \{i : \alpha + 1 \le i, 0 \le i \le n - d_k\}$, $k = 1,2$

as in [1]. Let $h_1 = \max\limits_{0\le i\le n-1}|x_{i+1} - x_i|$ and $h_2 = \max\limits_{0\le i\le n-1}|y_{i+1} - y_i|$. Since $d_1,d_2 > 0$, the following result can be obtained by [1]:

$$\left|\sum_{i=0}^{n-d_1}\sum_{j=0}^{n-d_2}\lambda_{i,j}(x,y)\right| = \frac{s(x,y)}{\prod\limits_{i=0}^{n}|x-x_i|\prod\limits_{i=0}^{n}|y-y_i|} \ge \frac{\mu_j(x)s(y)}{\prod\limits_{i=0}^{n}|x-x_i|\prod\limits_{i=0}^{n}|y-y_i|} \ge \frac{s(y)}{d_1!h_1^{d_1+1}\prod\limits_{i=0}^{n}|y-y_i|},\ j \in I_{2,1} \tag{13}$$

or

$$\left|\sum_{i=0}^{n-d_1}\sum_{j=0}^{n-d_2}\lambda_{i,j}(x,y)\right| = \frac{s(x,y)}{\prod\limits_{i=0}^{n}|x-x_i|\prod\limits_{i=0}^{n}|y-y_i|} \ge \frac{\mu_j(y)s(x)}{\prod\limits_{i=0}^{n}|x-x_i|\prod\limits_{i=0}^{n}|y-y_i|} \ge \frac{s(x)}{d_2!h_2^{d_2+1}\prod\limits_{i=0}^{n}|x-x_i|},\ j \in I_{2,2}. \tag{14}$$

Using the definition of μ_j yields

$$\frac{|\mu_i(x)|}{s(x)} \leq \frac{h_1}{t_1} \tag{15}$$

and

$$\frac{|\mu_i(y)|}{s(y)} \leq \frac{h_2}{t_2} \tag{16}$$

where $t_1 = \min\limits_{0\leq i\leq n-1} |x_{i+1}-x_i|$ and $t_2 = \min\limits_{0\leq i\leq n-1} |y_{i+1}-y_i|$. To see this, first let $x \in (x_\alpha, x_{\alpha+1})$ and $i \in I_{1,1}$. Then, $|\mu_i(x)| \leq \left|\mu_{\alpha-d_1}(x)\right|$. Since

$$\left|x - x_{\alpha-d_1}\right| \leq \frac{h_1}{t_1}\left|x - x_{\alpha+d_1+1}\right|,$$

we get the desired result. A similar argument can be made for $I_{3,1}$. We now deduce the following theorem for the $2-$dimensional case.

Theorem 8. *Suppose that* d_1, d_2 *are positive and* $f \in W_\infty^{n+1}(\prod\limits_{i=1}^{2}[0,b_i])$. *Let the interpolation nodes be* $\left\{(x_i,y_j) : 0 \leq i,j \leq n\right\}$. *Then,*

$$\|f-r\| \leq \frac{(n-d_1)(n-d_2)h_1^{d_1+1}}{t_1(d_1+1)}\left\|\frac{\partial^{d_1+1}f}{\partial x^{d_1+1}}\right\| + \frac{(n-d_1)(n-d_2)h_2^{d_2+1}}{t_2(d_2+1)}\left\|\frac{\partial^{d_2+1}f}{\partial y^{d_2+1}}\right\|$$
$$+\frac{(n-d_1)(n-d_2)h_1^{d_1+1}h_2^{d_2+1}}{(d_1+1)(d_2+1)}\left\|\frac{\partial^{d_1+d_2+2}f}{\partial x^{d_1+1}\partial y^{d_2+1}}\right\|$$

where

$$h_1 = \max_{0\leq i\leq n-1}|x_{i+1}-x_i|, h_2 = \max_{0\leq i\leq n-1}|y_{i+1}-y_i|,$$
$$t_1 = \min_{0\leq i\leq n-1}|x_{i+1}-x_i|,\ t_2 = \min_{0\leq i\leq n-1}|y_{i+1}-y_i|.$$

Proof. Since the error function $f-r$ is zero on the interpolation points, it is enough to find the error on the set

$$S := \left\{(x,y) : (x,y) \in [a_1,b_1]\times[a_2,b_2] \backslash \left\{(x_i,y_j) : 0 \leq i,j \leq n\right\}\right\}.$$

The function λ_{i_1,i_2} in (11) is well-defined on S and we can write the error function as

$$f(x,y)-r(x,y) = \frac{\sum\limits_{i_1=0}^{n-d_1}\sum\limits_{i_2=0}^{n-d_2}\lambda_{i_1,i_2}(x,y)\left[f(x,y)-p_{i_1,i_2}(x,y)\right]}{\sum\limits_{i_1=0}^{n-d_1}\sum\limits_{i_2=0}^{n-d_2}\lambda_{i_1,i_2}(x,y)}. \tag{17}$$

We will bound the error function by finding an upper bound on the numerator and a lower bound on the denominator of this quotient. The function $E = f - p_{i_1,i_2}f$ can be written by (9) as $E = E_1 + E_2 - E_1E_2$. Thus, the numerator of (17) is bounded by

$$|(f-r)(x,y)| \leq \frac{\frac{1}{(d_1+1)!}\sum\limits_{i_1=0}^{n-d_1}\sum\limits_{i_2=0}^{n-d_2}\frac{1}{\left|w_{i_2(y)}\right|}\left\|\frac{\partial^{d_1+1}f}{\partial x^{d_1+1}}\right\|}{\sum\limits_{i_1=0}^{n-d_1}\sum\limits_{i_2=0}^{n-d_2}\lambda_{i_1,i_2}(x,y)}$$
$$+\frac{\frac{1}{(d_2+1)!}\sum\limits_{i_1=0}^{n-d_1}\sum\limits_{i_2=0}^{n-d_2}\frac{1}{\left|w_{i_1(x)}\right|}\left\|\frac{\partial^{d_2+1}f}{\partial y^{d_2+1}}\right\|}{\sum\limits_{i_1=0}^{n-d_1}\sum\limits_{i_2=0}^{n-d_2}\lambda_{i_1,i_2}(x,y)}$$
$$+\frac{\frac{1}{(d_1+1)!(d_2+1)!}\sum\limits_{i_1=0}^{n-d_1}\sum\limits_{i_2=0}^{n-d_2}\left\|\frac{\partial^{d_1+d_2+2}f}{\partial x^{d_1+1}\partial y^{d_2+1}}\right\|}{\sum\limits_{i_1=0}^{n-d_1}\sum\limits_{i_2=0}^{n-d_2}\lambda_{i_1,i_2}(x,y)}.$$

We now obtain the inequality by applying (13-16)

$$
\begin{aligned}
|(f-r)(x,y)| \leq & \frac{\frac{1}{(d_1+1)!}\sum_{i_1=0}^{n-d_1}\sum_{i_2=0}^{n-d_2}\frac{1}{|w_{i_2(y)}|}\left\|\frac{\partial^{d_1+1}f}{\partial x^{d_1+1}}\right\|}{\frac{s(y)}{d_1!h_1^{d_1+1}\prod_{i=0}^{n}|y-y_i|}} \\
& + \frac{\frac{1}{(d_2+1)!}\sum_{i_1=0}^{n-d_1}\sum_{i_2=0}^{n-d_2}\frac{1}{|w_{i_1(x)}|}\left\|\frac{\partial^{d_2+1}f}{\partial y^{d_2+1}}\right\|}{\frac{s(x)}{d_2!h_2^{d_2+1}\prod_{i=0}^{n}|x-x_i|}} \\
& + \frac{\frac{1}{(d_1+1)!(d_2+1)!}\sum_{i_1=0}^{n-d_1}\sum_{i_2=0}^{n-d_2}\left\|\frac{\partial^{d_1+d_2+2}f}{\partial x^{d_1+1}\partial y^{d_2+1}}\right\|}{\frac{1}{d_1!h_1^{d_1+1}d_2!h_2^{d_2+1}}} \\
\leq & \frac{h_1^{d_1+1}}{d_1+1}\sum_{i_1=0}^{n-d_1}\sum_{i_2=0}^{n-d_2}\frac{|\mu_{i_2}(y)|}{s(y)}\left\|\frac{\partial^{d_1+1}f}{\partial x^{d_1+1}}\right\| \\
& + \frac{h_2^{d_2+1}}{d_2+1}\sum_{i_1=0}^{n-d_1}\sum_{i_2=0}^{n-d_2}\frac{|\mu_{i_2}(x)|}{s(x)}\left\|\frac{\partial^{d_2+1}f}{\partial y^{d_2+1}}\right\| \\
& + \frac{h_1^{d_1+1}h_2^{d_2+1}}{(d_1+1)(d_2+1)}\sum_{i_1=0}^{n-d_1}\sum_{i_2=0}^{n-d_2}\left\|\frac{\partial^{d_1+d_2+2}f}{\partial x^{d_1+1}\partial y^{d_2+1}}\right\| \\
\leq & (n-d_1)(n-d_2)\frac{h_1^{d_1+2}}{t_1(d_1+1)}\left\|\frac{\partial^{d_1+1}f}{\partial x^{d_1+1}}\right\| \\
& + (n-d_1)(n-d_2)\frac{h_2^{d_2+2}}{t_2(d_2+1)}\left\|\frac{\partial^{d_2+1}f}{\partial y^{d_2+1}}\right\| \\
& + \frac{(n-d_1)(n-d_2)h_1^{d_1+1}h_2^{d_2+1}}{(d_1+1)(d_2+1)}\left\|\frac{\partial^{d_1+d_2+2}f}{\partial x^{d_1+1}\partial y^{d_2+1}}\right\|.
\end{aligned}
$$

Simplifying the above inequality by using (14) yields the desired result.

Theorem 9. *Suppose that $d_1, d_2, \ldots, d_m\$$ are all positive and $f \in W_\infty^{n+1}\left(\prod_{i=0}^{m}[0,b_i]\right)$. Then*

$$\|f-r\| \leq \prod_{i=0}^{m}(n-d_i)\sum_{\|\alpha\|_\infty=1}\left(\frac{h}{t}\right)^\alpha h^{\alpha(d+1)}\left\|\partial^{\alpha(d+1)}f\right\|,$$

where

$$
\begin{aligned}
&\alpha(d+1) = (\alpha_1(d_1+1), \alpha_2(d_2+1), \ldots, \alpha_m(d_m+1)), \\
&h = (h_1, h_2, \ldots, h_m), t = (t_1, t_2, \ldots, t_m).
\end{aligned}
$$

Proof. It is proved by using the similar steps as in Theorem 8 and the inequalities

$$\frac{|\mu_{i_1,i_2,\ldots,i_m}(\mathbf{x})|}{s(\mathbf{x})} \leq 1,\ 0 \leq i_j \leq n-d_j,\ 0 \leq j \leq m,$$

where

$$s(\mathbf{x}) = \sum_{i=0}^{n-d_1}\mu_i(x_1)\sum_{i=0}^{n-d_2}\mu_i(x_2)\cdots\sum_{i=0}^{n-d_2}\mu_i(x_m).$$

4 Numerical Examples

In this section, several numerical examples are given to illustrate the properties and effectiveness of the method. We compare the approximate solution with polynomial interpolation and piecewise polynomial interpolation. All calculations were made using Maple 9.

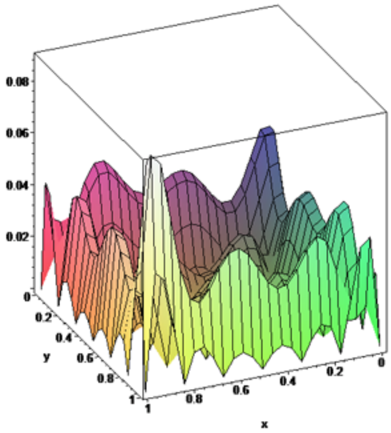

Fig. 1: Approximation error for $f(x,y) = 32(x+y)^{11/2}$ for $n = 6$ and $d_1 = d_2 = 4$.

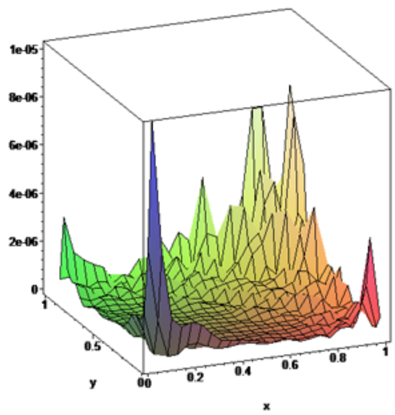

Fig. 2: Approximation error for $f(x,y) = 32(x+y)^{11/2}$ for $n = 10$ and $d_1 = d_2 = 7$.

Example 1. We want to approximate $f(x,y) = 32(x+y)^{\frac{11}{2}}$ on $[0,1]\times[0,1]$ which was given as an example by Mößner and Reif [8]. Selecting the nodes

$$\left\{(x_i,y_j) : x_i = \frac{i}{6},\ y_j = \frac{j}{6},\ 0 \le i,j \le 6\right\},$$

the absolute error for $n = 6$ and $d_1 = d_2 = 4$ is found as

$$\|f - r\|_\infty \le 0.09.$$

Similarly, taking the equidistant nodes, the absolute error for $n = 10$ and $d_1 = d_2 = 7$ is obtained as

$$\|f - r\|_\infty \le 10^{-5}.$$

The absolute errors for $n = 6$, $d_1 = d_2 = 4$ and $n = 10$, $d_1 = d_2 = 7$ are plotted in Figure 1 and Figure 2, respectively. Also, the infinity norms of error function for $n = 10$ and various d_1, d_2 are given in Table 1.

Table 1: The upper bounds of absolute errors for $n = 10$ and various d_1, d_2

$d_1 = d_2 = 3$	$d_1 = d_2 = 4$	$d_1 = d_2 = 5$	$d_1 = d_2 = 6$
0.031	0.007	$1.5E-4$	$4.4E-5$

Example 2. Let us consider the function $f(x,y) = x^2y^2\left(1 - e^{-(x^2+y^2)}\right)$ on $[0,1]\times[0,1]$ dealt with in Mößner& Reif [8]. Let us find the approximate solution for $n = 8$, $d_1 = d_2 = 5$ on equidistant nodes. After finding the approximate function r in (4), the upper bound of absolute error is found as

$$\|f - r\|_\infty \le 9.6 \times 10^{-7}.$$

The error function is plotted in Figure 3.

The following example was given in [9].

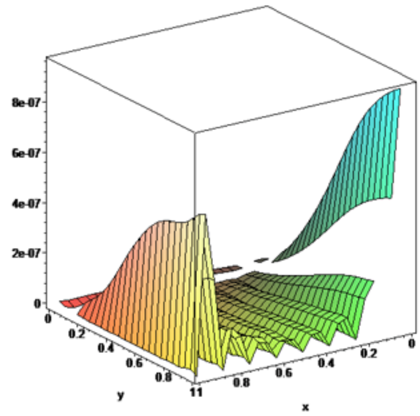

Fig. 3: Error function for $f(x,y) = x^2y^2\left(1-e^{-(x^2+y^2)}\right)$ on equidistant nodes.

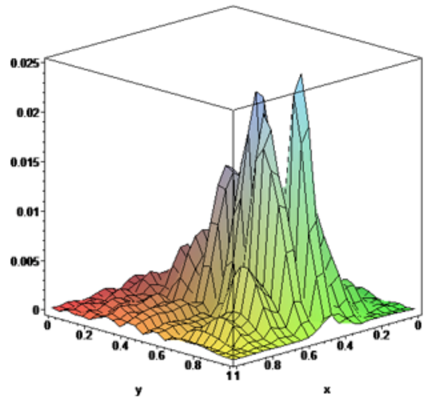

Fig. 4: Error function for $f(x,y) = \frac{3}{4}e^{-\frac{(9x-2)^2}{4}-\frac{(9y-2)^2}{4}} + \frac{3}{4}e^{-\frac{(9x-2)^2}{49}-\frac{(9y-2)^2}{10}} + \frac{1}{2}e^{-\frac{(9x-7)^2}{4}-\frac{(9y-3)^2}{4}} - \frac{1}{5}e^{-(9x-4)^2-(9y-7)^2}$.

We consider the function

$$f(x,y) = \frac{3}{4}e^{-\frac{(9x-2)^2}{4}-\frac{(9y-2)^2}{4}} + \frac{3}{4}e^{-\frac{(9x-2)^2}{49}-\frac{(9y-2)^2}{10}} + \frac{1}{2}e^{-\frac{(9x-7)^2}{4}-\frac{(9y-3)^2}{4}} - \frac{1}{5}e^{-(9x-4)^2-(9y-7)^2}$$

on $[0,1]\times[0,1]$. The upper bound of absolute error for $n = 10$, $d_1 = d_2 = 6$ is obtained as

$$\|f-r\|_\infty \le 0.035,$$

and the error function is plotted in Figure 4.

Example 3. We apply the method to

$$f(x,y) = e^x \cos y,\ (x,y) \in [0,1]\times[0,1]$$

which we sampled at the equidistant spaced points. The errors are given below for $n = 5$, $d_1 = d_2 = 4$and $n = 10$, $d_1 = d_2 = 7$:

$$\|f-r\|_\infty \le 0.9446\times 10^{-3}$$

and

$$\|f-r\|_\infty \le 0.3821\times 10^{-10}.$$

As a last example, to compare the approximate solution with piecewise polynomial interpolation, we give an example from [6].

Example 4. We compare the convergence of the piecewise polynomial interpolation of degree 2 and rational approximation for $n = 4$, $d_1 = d_2 = 2$ and $n = 8$, $d_1 = d_2 = 2$, on the function

$$f(x,y) = e^{-(x^2+y^2)}$$

on $[0,1]\times[0,1]$. While the errors, for $n = 4$, $d_1 = d_2 = 2$ with respect to piecewise polynomial interpolation and with respect to the present method are

$$\|f-r\|_\infty \le 1.6678\times 10^{-3}$$

and

$$\|f-r\|_\infty \leq 4.4842\times 10^{-3},$$

respectively, the errors, $n=8$, $d_1=d_2=2$, with respect to piecewise polynomial interpolation and with respect to the present method are

$$\|f-r\|_\infty \leq 2.8151\times 10^{-4}$$

and

$$\|f-r\|_\infty \leq 4.5000\times 10^{-5},$$

respectively.

5 Conclusion

Given any multivariate function f, one can approximate by (4) easily on a box and estimate its error by Theorem 9 provided that $f\in W_\infty^{n+1}\left(\prod_{i=0}^{m}[0,b_i]\right)$. Since the approximate solution (4) depends on polynomial interpolation, it may not converge. As seen from the examples, the method gives good approximation. More accurate results can be obtained for small d_i and n. If the function f is a polynomial, the method gives the exactly f since its interpolation polynomial is again itself. The method is applicable to all multivariate functions and it depends on $2\prod_{i=0}^{m}(n-d_i)$ function evaluates.

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