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Parallel surfaces of ruled Weingarten surfaces

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Received: 28 August 2015, Revised: 28 December 2015, Accepted: 29 December 2015
Published online: 31 December 2015.

Abstract: In this paper, it is shown that parallel surfaces of a non-developable ruled surface are not ruled surfaces by using fundamental forms. It has been shown that the parallel surfaces of a developable ruled surface is the developable ruled surfaces. It is obtained that parallel surfaces of ruled Weingarten surface are Weingarten surface.

Keywords: Parallel surface, Weingarten surfaces, ruled Weingarten surfaces

1 Introduction

Let f and g be smooth functions on a surface M in a Euclidean 3-space. The Jacobi function $\Phi(f,g)$ formed with f,g is defined by $\Phi(f,g) = \det\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$ where $f_u = \dfrac{\partial f}{\partial u}$ and $f_v = \dfrac{\partial f}{\partial v}$. In particular, a surface satisfying the Jacobi condition $\Phi(K,H) = 0$ with respect to the Gaussian curvature K and the mean curvature H is called a Weingarten surface or W-surface. All developable surfaces $(K = 0)$ and minimal surfaces $(H = 0)$ are Weingarten surfaces. Several geometers have studied Weingarten surfaces and obtained many interesting results [1-7]. In particular, Beltrami and Dini showed that the unique non-developable ruled Weingarten surface is a helicoidal ruled surface [1,3]. Later, in a different way, W. Kühnel gave a short proof for the old theorem by Beltrami and Dini on ruled Weingarten surfaces [5]. Also parallel surfaces are studied in many papers and obtained many interesting results [8,9].

2 Preliminaries

A (differentiable) one-parameter family of (straight) lines $\{\alpha(u), X(u)\}$ is a correspondence that assigns to each $u \in I$ a point $\alpha(u) \in \mathbb{R}^3$ and a vector $X(u) \in \mathbb{R}^3$, $X(u) \neq 0$, so that both $\alpha(u)$ and $X(u)$ are differentiable with respect to u. For each $u \in I$, the line L_u which passes through $\alpha(u)$ and parallel to $X(u)$ is called the line of the family at u.

Given a one-parameter family of lines $\{\alpha(u), X(u)\}$, the parametrized surface

$$\overrightarrow{\varphi}(u,v) = \alpha(u) + vX(u), \quad u \in I, \quad v \in \mathbb{R}$$

is called the ruled surface generated by the family $\{\alpha(u), X(u)\}$. The lines L_u are called the *rulings and the curve* $\alpha(u)$ is called a directrix of the surface φ. The normal vector of surface is denoted by $\overrightarrow{N}$. So the system $\{\alpha', X, N\}$ establishes an orthonormal frame along ruled surfaces.

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The parameter of distribution is expressed as follows

$$\lambda = \frac{\det(\alpha', X, X')}{|X'|^2}$$

where, as usual, (α', X, X') is a short for $\langle \alpha' \wedge X, X' \rangle$ [10].

Definition 1. *A ruled surface is called developable if the tangent planes to the surface are the same at all the points of a ruling [2].*

Proposition 1. *A ruled surface given by the equation*

$$\overrightarrow{\varphi}(u,v) = \alpha(u) + vX(u)$$

is developable if and only if

$$\det(\alpha', X, X') = 0 \tag{1}$$

The invariance condition for the tangent plane along a ruling is equivalent to the condition of invariance of the normal line to the surface along the same ruling. A normal vector to the surface is the vector

$$\overrightarrow{N}(u,v) = \alpha_u(u) \wedge X(u) + v\left[X_u(u) \wedge X(u)\right].$$

Thus, the condition for the surface to be developable is that the direction $\overrightarrow{N}$ be independent on the parameter v [11].

Definition 2. *A surface is called a Weingarten surface or W-surface if there is a nontrivial relation $\Phi(K,H) = 0$ or equivalently if the gradients of K and H are linearly dependent. In terms of the partial derivatives with respect to u and v this is the equation*

$$K_u H_v - K_v H_u = 0 \tag{2}$$

where K and H are Gaussian curvature and mean curvature of the surface [5].

A surface M^r whose points are at a constant distance along the normal from another surface M is said to be parallel to M. So, there are infinite number of such surfaces because we choose the constant distance along the normal arbitrarily. From the definition it follows that a parallel surface can be regarded as the locus of point which are on the normals to M at a non-zero constant distance r from M [8].

Definition 3. *Let M and M^r be two surfaces in Euclidean space. The function*

$$\begin{aligned} f : M &\to M^r \\ p &\to f(p) = p + r\overrightarrow{N}_p \end{aligned} \tag{3}$$

is called the parallelization function between M and M^r and furthermore M^r is called parallel surface to M where $\overrightarrow{N}$ is the unit normal vector field on M and r is a given real number [12].

Theorem 1. *Let M and M^r be two parallel surfaces in Euclidean space and*

$$f : M \to M^r \tag{4}$$

be the parallelization function. Then for $X \in \chi(M)$

(1) $f_*(X) = X + rS(X)$
(2) $S^r(f_*(X)) = S(X)$
(3) f *preserves principal directions of curvature, that is*

$$S^r(f_*(X)) = \frac{k}{1+rk} f_*(X)$$

where S^r *is the shape operator on* M^r*; and* k *is a principal curvature of* M *at* p *in direction* X *[12].*

Theorem 2. *The Gaussian curvature* K^r *and mean curvature* H^r *of parallel surface* M^r *are respectively*

$$K^r = \frac{K}{1+rH+r^2K} \quad and \quad H^r = \frac{H+2rK}{1+rH+r^2K} \tag{5}$$

where K *and* H *are Gaussian curvature and mean curvature of surface* M *respectively [12].*

3 Parallel surfaces of ruled surfaces

Let $\overrightarrow{\varphi}$ be the position vector of a point P on M. Therefore the parametrization for M is given by

$$\overrightarrow{\varphi}(u,v) = \alpha(u) + vX(u). \tag{6}$$

By differentiating $\overrightarrow{\varphi}(u,v)$ with respect to u and v respectively, we get

$$\overrightarrow{\varphi}_u(u,v) = \alpha_u(u) + vX_u(u) \text{ and } \overrightarrow{\varphi}_v(u,v) = X(u).$$

The normal vector of surface M is given as follows:

$$\overrightarrow{N}(u,v) = \alpha_u(u) \wedge X(u) + v(X_u(u) \wedge X(u)). \tag{7}$$

By differentiating $\overrightarrow{N}(u,v)$ with respect to u and v respectively, the following expressions are obtained:

$$\begin{aligned} \overrightarrow{N}_u &= \alpha_{uu} \wedge X + \alpha_u \wedge X_u + v(X_{uu} \wedge X) \\ \overrightarrow{N}_v &= X_u \wedge X \end{aligned}$$

The coefficients of first and second fundamental forms of M are given by

$$\begin{aligned} \mathbf{E} &= <\overrightarrow{\varphi}_u, \overrightarrow{\varphi}_u> = <\alpha_u, \alpha_u> + 2v <\alpha_u, X_u> + v^2 <X_u, X_u> \\ \mathbf{F} &= <\overrightarrow{\varphi}_u, \overrightarrow{\varphi}_v> = <\alpha_u, X> + v <X_u, X> \\ \mathbf{G} &= <\overrightarrow{\varphi}_v, \overrightarrow{\varphi}_v> = <X, X> \end{aligned} \tag{8}$$

and

$$\begin{aligned} \mathbf{L} &= -<\overrightarrow{\varphi}_u, \overrightarrow{N_u}> \\ &= -(<\alpha_u, \alpha_{uu} \wedge X> + v <\alpha_u, X_{uu} \wedge X> \\ &\quad + v <X_u, \alpha_{uu} \wedge X> + v^2 <X_u, X_{uu} \wedge X>) \\ \mathbf{M} &= -<\overrightarrow{\varphi}_u, \overrightarrow{N_v}> = -<\alpha_u, X_u \wedge X> \\ \mathbf{N} &= -<\overrightarrow{\varphi}_v, \overrightarrow{N_v}> = -<X, X_u \wedge X> = 0 \end{aligned} \tag{9}$$

The coefficient **N** as the last coefficient of second fundamental form of all ruled surfaces always zero. The inner product of vector $\overrightarrow{N}_u$ with itself is calculated as follows;

$$\begin{aligned} <\overrightarrow{N}_u,\overrightarrow{N}_u> = &<\alpha_{uu}\wedge X,\alpha_{uu}\wedge X> +2<\alpha_{uu}\wedge X,\alpha_u\wedge X_u> \\ &+2v<\alpha_{uu}\wedge X,X_{uu}\wedge X> + <\alpha_u\wedge X_u,\alpha_u\wedge X_u> \\ &+2v<\alpha_u\wedge X_u,X_{uu}\wedge X> +v^2<X_{uu}\wedge X,X_{uu}\wedge X> \end{aligned} \tag{10}$$

and the inner product of $\overrightarrow{N}_u$ and $\overrightarrow{N}_v$ is

$$\begin{aligned} <\overrightarrow{N}_u,\overrightarrow{N}_v> = &<\alpha_{uu}\wedge X,X_u\wedge X> + <\alpha_u\wedge X_u,X_u\wedge X> \\ &+v<X_{uu}\wedge X,X_u\wedge X> \end{aligned} \tag{11}$$

and last, the inner product of vector $\overrightarrow{N}_v$ with itself is found as follows

$$<\overrightarrow{N}_v,\overrightarrow{N}_v>=<X_u\wedge X,X_u\wedge X>. \tag{12}$$

3.1 Fundamental forms of parallel surfaces of a ruled surface

First, we obtain the representation of points on M^r using the representations of points on M. Let $\overrightarrow{\varphi}$ be the position vector of a point P on M and $\overrightarrow{\varphi}^r$ be the position vector of a point P^r on the parallel surface M^r. Then P^r is at a constant distance r from P along normal to the surface M. Therefore the parametrization for parallel surface M^r is given by

$$\begin{aligned} \overrightarrow{\varphi}^r(u,v) &= \overrightarrow{\varphi}(u,v)+r\overrightarrow{N}(u,v) \\ &= \alpha(u)+vX(u)+r(\alpha_u(u)\wedge X(u)+v(X_u(u)\wedge X(u))) \\ &= \alpha(u)+r(\alpha_u(u)\wedge X(u))+v[X(u)+r(X_u(u)\wedge X(u))] \end{aligned} \tag{13}$$

where r is constant scalar and $\overrightarrow{N}$ is the normal vector field on M.

Theorem 3. *The coefficients of first and second fundamental forms of the parallel surface M^r are obtained by*

$$\begin{aligned} \mathbf{E}^r &= E-2rL+r^2<\overrightarrow{N}_u,\overrightarrow{N}_u> \\ \mathbf{F}^r &= F-2rM+r^2<\overrightarrow{N}_u,\overrightarrow{N}_v> \\ \mathbf{G}^r &= G+r^2<\overrightarrow{N}_v,\overrightarrow{N}_v> \end{aligned} \tag{14}$$

and

$$\begin{aligned} \mathbf{L}^r &= L-r<\overrightarrow{N}_u,\overrightarrow{N}_u> \\ \mathbf{M}^r &= M-r<\overrightarrow{N}_u,\overrightarrow{N}_v> \\ \mathbf{N}^r &= -r<\overrightarrow{N}_v,\overrightarrow{N}_v> \end{aligned} \tag{15}$$

Proof. The coefficients of first fundamental form are calculated as follows;

$$\begin{aligned} \mathbf{E}^r &= <\overrightarrow{\varphi}^r_u,\overrightarrow{\varphi}^r_u> \\ &= <\alpha_u+r(\alpha_{uu}\wedge X+\alpha_u\wedge X_u)+v[X_u+r(X_{uu}\wedge X)], \\ &\qquad \alpha_u+r(\alpha_{uu}\wedge X+\alpha_u\wedge X_u)+v[X_u+r(X_{uu}\wedge X)]> \end{aligned}$$

By using the equations (8), (9) and (10), the following equation is obtained

$$\mathbf{E}^r = \mathbf{E}-2r\mathbf{L}+r^2<N_u,N_u>. \tag{16}$$

Calculation of the coefficient $\mathbf{F}^r$ is obtained as follows

$$\begin{aligned}\mathbf{F}^r &= <\overrightarrow{\varphi}^r_u, \overrightarrow{\varphi}^r_v> \\ &= < \alpha_u + r\left(\alpha_{uu} \wedge X + \alpha_u \wedge X_u\right) + v\left[X_u + r\left(X_{uu} \wedge X\right)\right], X + r(X_u \wedge X) >\end{aligned}$$

By using the equations (8), (9) and (11), the following equation is obtained

$$\mathbf{F}^r = \mathbf{F} - 2r\mathbf{M} + r^2 < \overrightarrow{N}_u, \overrightarrow{N}_v > . \tag{17}$$

And calculation of the coefficient $\mathbf{G}^r$ is given as follows

$$\begin{aligned}\mathbf{G}^r &= < \overrightarrow{\varphi}^r_v, \overrightarrow{\varphi}^r_v > \\ &= < X + r(X_u \wedge X), X + r(X_u \wedge X) >\end{aligned}$$

From the equations (8) and (12), the following equation is given

$$\mathbf{G}^r = \mathbf{G} + r^2 < \overrightarrow{N}_v, \overrightarrow{N}_v > . \tag{18}$$

The coefficients of second fundamental form are obtained

$$\begin{aligned}\mathbf{L}^r &= - < \overrightarrow{\varphi}^r_u, \overrightarrow{N}_u > \\ &= - < \alpha_u + r\left(\alpha_{uu} \wedge X + \alpha_u \wedge X_u\right) + v\left[X_u + r\left(X_{uu} \wedge X\right)\right], \\ &\qquad \alpha_{uu} \wedge X + \alpha_u \wedge X_u + v\left(X_{uu} \wedge X\right) >\end{aligned}$$

By using the equations (9) and (10), the following equation is found

$$\mathbf{L}^r = \mathbf{L} - r<\overrightarrow{N}_u, \overrightarrow{N}_u> . \tag{19}$$

Calculation of the coefficient $\mathbf{M}^r$

$$\begin{aligned}\mathbf{M}^r &= - < \overrightarrow{\varphi}^r_u, \overrightarrow{N}_v > \\ &= - < \alpha_u + r\left(\alpha_{uu} \wedge X + \alpha_u \wedge X_u\right) + v\left[X_u + r\left(X_{uu} \wedge X\right)\right], X_u \wedge X >\end{aligned}$$

or

$$\begin{aligned}\mathbf{M}^r &= - < \overrightarrow{\varphi}^r_v, \overrightarrow{N}_u > \\ &= - < X + r(X_u \wedge X), \overrightarrow{\alpha}_{uu} \wedge \overrightarrow{X} + \overrightarrow{\alpha}_u \wedge \overrightarrow{X}_u + v\left(\overrightarrow{X}_{uu} \wedge \overrightarrow{X}\right) >\end{aligned}$$

By using the equations (9) and (11), the following equation is obtained

$$\mathbf{M}^r = \mathbf{M} - r < \overrightarrow{N}_u, \overrightarrow{N}_v > . \tag{20}$$

And last, calculation of the coefficient $\mathbf{N}^r$

$$\begin{aligned}\mathbf{N}^r &= - < \overrightarrow{\varphi}^r_v, \overrightarrow{N}_v > \\ &= - < X + r(X_u \wedge X), X_u \wedge X >\end{aligned}$$

By using the equation (12), we have

$$\mathbf{N}^r = -r < \overrightarrow{N}_v, \overrightarrow{N}_v > . \tag{21}$$

So, for a parallel surface of a ruled surface to become a ruled surface, $\overrightarrow{N}_v = 0$, i.e. the normal vector of surface has to be independ of parameter v.

3.2 Parallel developable ruled surfaces

Theorem 4. *Parallel surface of a developable ruled surface is also a developable ruled surface.*

Proof. Let $\overrightarrow{\varphi}(u,v) = \alpha(u) + vX(u)$ be developable ruled surface such that $\|X(u)\| = 1$ and $\|X_u(u)\| = 1$. We have the normal vector of surface $\overrightarrow{\varphi}(u,v)$ as follows:

$$\overrightarrow{N}_1 = \alpha_u(u) \wedge X(u) + v(X_u(u) \wedge X(u)). \tag{22}$$

From the Proposition 1, we have normal vector of the developable ruled surface is independent from parameter v. Hence the expressions $\alpha_u(u) \wedge X(u)$ and $X_u(u) \wedge X(u)$ in (22) are linearly dependent, that is,

$$\alpha_u(u) \wedge X(u) = \varepsilon(X_u(u) \wedge X(u)).$$

So, the unit normal vector of the main surface becomes

$$\overrightarrow{N} = X_u \wedge X. \tag{23}$$

Using the equation (23) the parallel surface of the developable ruled surface $\overrightarrow{\varphi}(u,v)$ parameterized as

$$\overrightarrow{\varphi}^r(u,v) = \alpha(u) + r(X_u(u) \wedge X(u)) + vX(u).$$

If the equation (2) is calculated as regarding with the parallel ruled surface $\overrightarrow{\varphi}^r(u,v)$

$$\begin{aligned} \lambda &= < \alpha_u + rX_{uu} \wedge X, X_u \wedge X > \\ &= < \alpha_u, X_u \wedge X > + r < X_{uu} \wedge X, X_u \wedge X >, \end{aligned} \tag{24}$$

if $< X_u, X_u >= 1$ then $< X_u, X_{uu} >= 0$. Then the following expression

$$X_{uu} = aX + bX_u \wedge X$$

can be written. Hence

$$\begin{aligned} X_{uu} \wedge X &= (aX + bX_u \wedge X) \wedge X \\ &= aX \wedge X + b(X_u \wedge X) \wedge X \\ &= -bX_u \end{aligned} \tag{25}$$

is obtained. Since $det(\alpha_u, X_u, X)$ is zero by th equation (1), and also by using the equation (25) the following result is obtained:

$$\begin{aligned} \lambda &= < \alpha_u, X_u \wedge X > + r < X_{uu} \wedge X, X_u \wedge X > \\ &= -rb < X_u, X_u \wedge X > \\ &= 0 \end{aligned}$$

Hence $\overrightarrow{\varphi}^r(u,v)$ is a developable ruled surface. The coefficients of second fundamental form of the parallel ruled surface are found

$$\begin{aligned}\mathbf{L}^r &= -r < \overrightarrow{N}_u, \overrightarrow{N}_u > \\ \mathbf{M}^r &= 0 \\ \mathbf{N}^r &= 0\end{aligned}$$

Thus the parallel surface of a developable ruled surface is a developable ruled surface, its directrix curve $\alpha(u) + r(X_u(u) \wedge X(u))$ and its ruling $X(u)$.

4 Parallel surfaces of ruled Weingarten surfaces

Theorem 5. *Let M^r be a parallel surface of a surface M in Euclidean 3-space. If M^r is a Weingarten surface then M is a Weingarten surface.*

Proof. From the equation (5) we have

$$\begin{aligned}&K_u^r H_v^r - K_v^r H_u^r \\ &= \left(\frac{K}{1+rH+r^2K}\right)_u \left(\frac{H+2rK}{1+rH+r^2K}\right)_v - \left(\frac{K}{1+rH+r^2K}\right)_v \left(\frac{H+2rK}{1+rH+r^2K}\right)_u\end{aligned} \tag{26}$$

By differentiating the expression (26) with respect to the parameters u and v, the following equation

$$K_u^r H_v^r - K_v^r H_u^r = \frac{1}{(1+rH+r^2K)^3}(K_u H_v - K_v H_u) \tag{27}$$

is obtained. From the equation (5) and (27), the parallel surfaces of a ruled Weingarten surface are also Weingarten surfaces.

Example 1. The helicoid surface given by the parameterization $\overrightarrow{\varphi}(u,v) = (v\cos u, v\sin u, au)$ determines a ruled surface shown as

$$\begin{aligned}\overrightarrow{\varphi}(u,v) &= (0,0,au) + v(\cos u, \sin u, 0) \\ &= \alpha(u) + vX(u).\end{aligned} \tag{28}$$

Let's show that this ruled surface and its parallel surface is Weingarten surface. Let's differentiate this surface $\varphi(u,v)$ with respect to the parameters u and v in order to find this surface's fundamental forms, so

$$\begin{aligned}\overrightarrow{\varphi}_u &= (0,0,a) + v(-\sin u, \cos u, 0) \\ \overrightarrow{\varphi}_v &= (\cos u, \sin u, 0) \\ \overrightarrow{\varphi}_{uu} &= v(-\cos u, -\sin u, 0) \\ \overrightarrow{\varphi}_{uv} &= (-\sin u, \cos u, 0) \\ \overrightarrow{\varphi}_{vv} &= 0.\end{aligned}$$

The unit normal vector of this surface is obtained as

$$\overrightarrow{N} = \frac{\overrightarrow{\varphi}_u \wedge \overrightarrow{\varphi}_v}{\|\overrightarrow{\varphi}_u \wedge \overrightarrow{\varphi}_v\|} = \frac{1}{\sqrt{a^2+v^2}}(-a\sin u, a\cos u, v) \tag{29}$$

Thus coefficients of the first fundamental form of the surface

$$\begin{aligned}\mathbf{E} &= < \overrightarrow{\varphi}_u, \overrightarrow{\varphi}_u > = a^2+v^2 \\ \mathbf{F} &= < \overrightarrow{\varphi}_u, \overrightarrow{\varphi}_v > = 0 \\ \mathbf{G} &= < \overrightarrow{\varphi}_v, \overrightarrow{\varphi}_v > = 1\end{aligned} \tag{30}$$

coefficients of the second fundamental form

$$\begin{aligned}\mathbf{L} &= < \overrightarrow{\varphi}_{uu}, \overrightarrow{N} > = 0 \\ \mathbf{M} &= < \overrightarrow{\varphi}_{uv}, \overrightarrow{N} > = \frac{a}{\sqrt{a^2+v^2}} \\ \mathbf{N} &= < \overrightarrow{\varphi}_{vv}, \overrightarrow{N} > = 0\end{aligned} \tag{31}$$

By using these coefficients found above, the Gauss and mean curvatures of the surface are found as follows

$$K = -\frac{a^2}{(a^2+v^2)^2} \quad \text{and} \quad H = 0. \tag{32}$$

By calculating the partial derivatives of the curvatures K and H with rescept to u and v then substituting them into the equation (2), the following expression is found

$$K_u H_v - K_v H_u = 0.$$

Thus helicoid surface is a Weingarten surface. Let's show that parallel surface of this helicoid surface is again a Weingarten surface. From the equation (32), Gaussian and mean curvatures of parallel surface are found by substituting the values of the curvatures K and H into the equation (5):

$$K^r = -\frac{a^2}{(a^2+v^2)^2 - r^2a^2} \quad \text{and} \quad H^r = -\frac{2ra^2}{(a^2+v^2)^2 - r^2a^2}$$

are obtained. Since the derivatives of the curvatures K^r and H^r with respect to u are zero,

$$K_u^r H_v^r - K_v^r H_u^r = 0$$

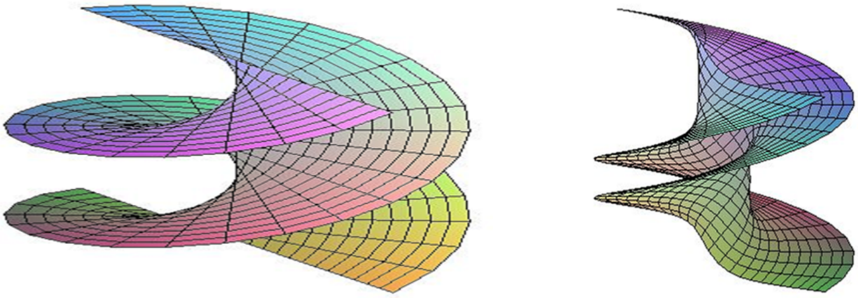
becomes. Then parallel surface of helicoidal ruled Weingarten surface is again a Weingarten surface. Helicoidal ruled Weingarten surface given by the following parameterization

$$\overrightarrow{\varphi}(u,v) = (v\cos u, v\sin u, au)$$

And its parallel surface as a Weingarten surface has the following parameterized equation

$$\overrightarrow{\varphi}^r(u,v) = (v\cos u - \frac{1}{\sqrt{a^2+v^2}}a\sin u, v\sin u + \frac{1}{\sqrt{a^2+v^2}}a\cos u, au+v).$$

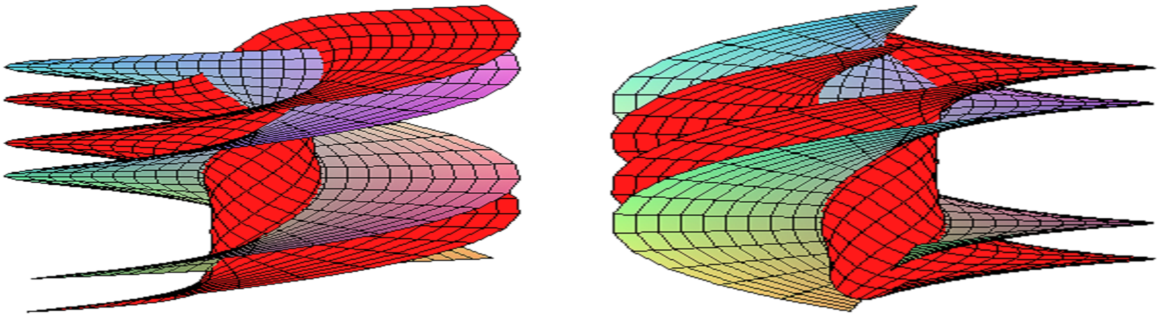
Ruled Weingarten surface and its parallel surface are, respectively, seen in the following figure 1(a) and figure 1 (b),



(a) (b)

Fig. 1: Ruled Weingarten surface and its parallel surface.

The following figures present these above surfaces together, so the red surface represent the parallel surface and the other represent ruled Weingarten surface in figure 2(a) and figure 2(b),



(a) (b)

Fig. 2: The red surface represent the parallel surface and the other represent ruled Weingarten surface.

5 Conclusion

In this paper, we show that parallel surfaces of non-developable ruled surfaces are not ruled surfaces via using fundamental forms. Also we study parallel surfaces of developable ruled surfaces. Finally, we proved that parallel surfaces of ruled Weingarten surface are also Weingarten surfaces.

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