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Inextensible flows of curves in the equiform geometry of the double isotropic space $\mathscr{I}_3^{(2)}$

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Abstract: The evolution equations are derived for double isotropic inextensible flows of curves in the equiform geometry of the double isotropic space $\mathscr{I}_3^{(2)}$.

Keywords: Double isotropic, inextensible flow, equiform geometry

1 Introduction

The flow of a curve in R^3 is called inextensible if, its arclength is preserved. Inextensible curve flows give rise to when strain energy is not exist in motion [1]. Kwon and Park examine inextensible flow of curves and developable surfaces for plane and space curves [1-2].

Inextensible curves has many applications in computer vision, snake-like, robots [3-4]. Chirikjian, Burdick, Mochiyama research the shape control of hyper-redundant, snake-like, robots. They present new and efficient kinematic methods that are suitable for nearly all hyper-redundant robot morphologies using motion of curves [5]. Hamilton, Gage and Grayson study shrinking of closed plane curves to a circle via the heat equation [6-8].

Gurbuz investigated inextensible flow of non-null ve null curves in Minkowski 3-space [9]. Bektas and Kulahci studied inextensible curves in E_1^4 [10]. Yildiz, Tosun, Ozkaldi, Karakus derived inextensible flows of curves in E^n [11],[12]. Gurbuz studied inextensible flows of non-null curves on an pseudo-Euclidean hypersuface in pseudo-Euclidean space R_1^n [13]. Yoon studied inelastic flows of curves according to equiform geometry in Galilean space [14].

In this work, the evolution equations are derived for double isotropic inextensible flows of curves in the equiform geometry of the double isotropic space $\mathscr{I}_3^{(2)}$. Necessary and sufficient conditions are expressed for double isotropic inextensible curve flows in the equiform geometry of the double isotropic space $\mathscr{I}_3^{(2)}$.

2 Preliminaries

The double isotropic geometry is one of the real Cayley-Klein geometries. The equiform differential geometry of the double isotropic space $\mathscr{I}_3^{(2)}$ has been investigated in detail [15],[16]. The scalar product of two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathscr{I}_3^{(2)}$ is defined by

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$$\langle a,b\rangle|_{\mathscr{I}_3^{(2)}} = \begin{cases} a_1b_1, & a_1 \neq 0 \text{ or } b_1 \neq 0 \\ a_2b_2, & a_1 = b_1 = a_2 = b_2 = 0 \end{cases}$$

The equiform curvature and the equiform torsion of an admissible curve is defined by [17]

$$\alpha : \mathscr{I} \subset \mathscr{R} \to \mathscr{I}_3^{(2)}$$

parametrized by the arc of length $ds = dx$, given by $c(s) = (x, y(x), z(x))$. The curvature $\kappa(s)$ and the torsion is $\tau(s)$ are defined by

$$\kappa(x) = y''(x),\ \ \tau(x) = (\frac{z''(x)}{y''(x)})'.$$

The equiform curvature and the equiform torsion of an admissible curve are defined by

$$\widehat{\kappa} = \dot{\rho}, \qquad \widehat{\tau} = \rho\tau = \tfrac{\tau}{\kappa}$$

where ρ is the radius of curvature of the curve α The associated trihedron is expressed

$$\widehat{T} = \rho t, \qquad \widehat{N} = \rho n, \qquad \widehat{B} = \rho b$$

The formulas analogous to the Frenet's frame in the equiform geometry of the double isotropic space $\mathscr{I}_3^{(2)}$ have the following form [16] :

$$\begin{aligned} \frac{d\widehat{T}}{d\sigma} &= \widehat{\kappa}\widehat{T} + \widehat{N}, \\ \frac{d\widehat{N}}{d\sigma} &= \widehat{\kappa}\widehat{N} + \widehat{\tau}\widehat{B}, \\ \frac{d\widehat{B}}{d\sigma} &= \widehat{\kappa}\widehat{B}, \end{aligned} \tag{1}$$

where σ is an equiform invariant parameter by $\sigma = \int \dfrac{ds}{\rho}$.

3 Inextensible flows of curves in the equiform geometry of the double isometric space $\mathscr{I}_3^{(2)}$

Let $\Psi : [0,l] \times [0,t) \to \mathscr{I}_3^{(2)}$ be a family of differentiable curves in the equiform geometry of the double isometric space $\mathscr{I}_3^{(2)}$. Let w be the curve parametrization variable and the curve speed $v = \left\| \frac{\partial \Psi}{\partial w} \right\|$,the arclength of Ψ is defined by

$$s(w) = \int v dw.$$

For the orthonormal frame $\left\{ \widehat{T}, \widehat{N}, \widehat{B} \right\}$ in the equiform geometry of a curve β in double isotropic space $\mathscr{I}_3^{(2)}$.

Any flow of Ψ is given by

$$\frac{\partial \Psi}{\partial t} = \lambda \widehat{T} + \gamma \widehat{N} + \eta \widehat{B}.$$

Here λ, γ, η are differentiable functions.

In the equiform geometry of double isotropic $\mathscr{I}_3^{(2)}$, a curve evolution $\Psi(\omega, t)$ and its flow $\frac{\partial \Psi}{\partial t}$ is called to be double isotropic inextensible if

$$\frac{\partial}{\partial t}\sqrt{\left\langle \frac{\partial \Psi}{\partial \omega}, \frac{\partial \Psi}{\partial \omega}\right\rangle\Big|_{\mathscr{I}_3^{(2)}}} = 0. \tag{2}$$

Theorem 1.

$$\frac{\partial v}{\partial t} = \frac{\partial \lambda}{\partial \omega} + v\lambda\widehat{\kappa} \tag{3}$$

Proof.

$$v^2 = \left\langle \frac{\partial \Psi}{\partial \omega}, \frac{\partial \Psi}{\partial \omega}\right\rangle\Big|_{\mathscr{I}_3^{(2)}}.$$

It can be written

$$\begin{aligned} 2v\frac{\partial v}{\partial t} =& \frac{\partial}{\partial t}\left\langle \frac{\partial \Psi}{\partial \omega}, \frac{\partial \Psi}{\partial \omega}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} = 2\left\langle \frac{\partial \Psi}{\partial \omega}, \frac{\partial}{\partial \omega}(\lambda\widehat{T} + \gamma\widehat{N} + \eta\widehat{B})\right\rangle\Big|_{\mathscr{I}_3^{(2)}} \\ =& 2\left\langle \begin{array}{c} v\widehat{T}, (\frac{\partial \lambda}{\partial \omega} + v\lambda\widehat{\kappa})\widehat{T} + (\lambda v + \frac{\partial \gamma}{\partial \omega} + \gamma v\widehat{\kappa})\widehat{N} \\ +(v\gamma\widehat{\tau} + \frac{\partial \eta}{\partial \omega} + \eta v\widehat{\kappa})\widehat{B} \end{array}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} \\ =& 2v(\frac{\partial \lambda}{\partial \omega} + v\lambda\widehat{\kappa}). \end{aligned} \tag{4}$$

From the Theorem (1), it is obtained

$$\frac{\partial v}{\partial t} = \frac{\partial \lambda}{\partial \omega} + v\lambda\widehat{\kappa} \tag{5}$$

Theorem 2. *Let* $\frac{\partial \Psi}{\partial t} = \lambda\widehat{T} + \gamma\widehat{N} + \eta\widehat{B}$ *be a differentiable flow in the equiform geometry of double isotropic space* $\mathscr{I}_3^{(2)}$*. The curve flow is double isotropic inextensible if and only*

$$\frac{\partial \lambda}{\partial s} = -\lambda\widehat{\kappa}. \tag{6}$$

Proof. From the Theorem (1) and (5), it can be obtained

$$\frac{\partial}{\partial t}s(w,t) = \int \frac{\partial v}{\partial t}du = \int (\frac{\partial \lambda}{\partial \omega} + v\lambda\widehat{\kappa})dw = 0. \tag{7}$$

With aid (7), we obtain (6).

Corollary 1. *A curve flow is independent of components normal* γ *and binormal* η *in the equiform geometry of the double isotropic space* $\mathscr{I}_3^{(2)}$*.*

Theorem 3. *Let* $\left\{\widehat{T}, \widehat{N}, \widehat{B}\right\}$ *be the Frenet frame in the equiform geometry of double isotropic space. Then*

$$\begin{aligned} \frac{\partial \widehat{T}}{\partial t} =& (\lambda + \frac{\partial \gamma}{\partial s} + \gamma\widehat{\kappa})\widehat{N} + (\gamma\widehat{\tau} + \frac{\partial \eta}{\partial s} + \eta\widehat{\kappa})\widehat{B} \\ \frac{\partial \widehat{N}}{\partial t} =& -(\lambda + \frac{\partial \gamma}{\partial s} + \gamma\widehat{\kappa})\widehat{T} + \Omega\widehat{B}, \\ \frac{\partial \widehat{B}}{\partial t} =& -(\gamma\widehat{\tau} + \frac{\partial \eta}{\partial s} + \eta\widehat{\kappa})\widehat{T} - \Omega\widehat{N}. \end{aligned}$$

Proof.

$$\begin{aligned}\frac{\partial \widehat{T}}{\partial t} =& \frac{\partial}{\partial t}\frac{\partial \Psi}{\partial s} = \frac{\partial}{\partial s}\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial s}(\lambda \widehat{T} + \gamma \widehat{N} + \eta \widehat{B}) \\ =& (\frac{\partial \lambda}{\partial s} + \lambda \widehat{\kappa})\widehat{T} + (\lambda + \frac{\partial \gamma}{\partial s} + \gamma \widehat{\kappa})\widehat{N} + (\gamma \widehat{\tau} + \frac{\partial \eta}{\partial s} + \eta \widehat{\kappa})\widehat{B}.\end{aligned} \tag{8}$$

From Theorem (3.1), (6), (8) we obtain

$$\frac{\partial \widehat{T}}{\partial t} = (\lambda + \frac{\partial \gamma}{\partial s} + \gamma \widehat{\kappa})\widehat{N} + (\gamma \widehat{\tau} + \frac{\partial \eta}{\partial s} + \eta \widehat{\kappa})\widehat{B}. \tag{9}$$

Using the Frenet frame in the equiform geometry of double isotropic space, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle \widehat{T}, \widehat{N}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} =& \left\langle \frac{\partial \widehat{T}}{\partial t}, \widehat{N}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} + \left\langle \frac{\partial \widehat{N}}{\partial t}, \widehat{T}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} \\ =& \lambda \widehat{\kappa} + \frac{\partial \gamma}{\partial s} + \eta \widehat{\tau} + \left\langle \frac{\partial \widehat{N}}{\partial t}, \widehat{T}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} = 0\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle \widehat{T}, \widehat{B}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} =& \left\langle \frac{\partial \widehat{T}}{\partial t}, \widehat{B}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} + \left\langle \widehat{T}, \frac{\partial \widehat{B}}{\partial t}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} \\ =& (\gamma \widehat{\tau} + \frac{\partial \eta}{\partial s} + \eta \widehat{\kappa}) + \left\langle \frac{\partial \widehat{B}}{\partial t}, \widehat{T}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} = 0\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle \widehat{N}, \widehat{B}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} =& \left\langle \frac{\partial \widehat{N}}{\partial t}, \mathscr{B}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} + \left\langle \frac{\partial \widehat{B}}{\partial t}, \widehat{N}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} \\ =& \Omega + \left\langle \frac{\partial \widehat{B}}{\partial t}, \widehat{N}\right\rangle\Big|_{\mathscr{I}_3^{(2)}} = 0\end{aligned} \tag{12}$$

Using (9), (10), (11), (12), we obtain

$$\frac{\partial \widehat{N}}{\partial t} = -(\lambda \widehat{\kappa} + \frac{\partial \gamma}{\partial s} + \eta \widehat{\tau})\widehat{T} + \Omega \widehat{B}, \tag{13}$$

$$\frac{\partial \widehat{B}}{\partial t} = -(\gamma \widehat{\tau} + \frac{\partial \eta}{\partial s} + \eta \widehat{\kappa})\widehat{T} - \Omega \widehat{N}. \tag{14}$$

We give main theorem. This theorem gives necessary and sufficient conditions for a double inextensible curve flow

Theorem 4. *Assume the curve flow* $\frac{\partial \Psi}{\partial t} = \lambda \widehat{T} + \gamma \widehat{N} + \eta \widehat{B}$ *is double isotropic inextensible in the equiform geometry of the double isotropic space. Evolution of double isotropic curvature and torsion are expressed as a system of partial differential equations:*

$$\frac{\partial \widehat{\kappa}}{\partial t} = -(\frac{\partial^2 \eta}{\partial s^2} + 3\widehat{\tau}\frac{\partial \gamma}{\partial s} + \frac{\partial \widehat{\kappa}}{\partial s} + \widehat{\tau}\lambda + \widehat{\kappa}\lambda\widehat{\tau})\widehat{\tau},$$

$$\frac{\partial \widehat{\tau}}{\partial t} = \frac{\partial \Omega}{\partial s},$$

$$\Omega = \frac{\partial^2\eta}{\partial s^2} + 3\widehat{\tau}\frac{\partial\gamma}{\partial s} + \frac{\partial\widehat{\kappa}}{\partial s} + \widehat{\tau}\lambda + \widehat{\kappa}\lambda\widehat{\tau}. \tag{15}$$

Proof. Using (9) and the formulas analogous to the Frenet's frame in the equiform geometry of the double isotropic space

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial\widehat{T}}{\partial t} =& \frac{\partial}{\partial s}(\lambda + \frac{\partial\gamma}{\partial s} + \gamma\widehat{\kappa})\widehat{N} + (\gamma\widehat{\tau} + \frac{\partial\eta}{\partial s} + \eta\widehat{\kappa})\widehat{B}\\ =& (\frac{\partial^2\eta}{\partial s^2} + 2\widehat{\kappa}\frac{\partial\gamma}{\partial s} + \gamma\frac{\partial\widehat{\kappa}}{\partial s} + \gamma\widehat{\kappa}^2)\widehat{N} +\\ & (2\widehat{\tau}\frac{\partial\gamma}{\partial s} + 2\gamma\widehat{\kappa}\widehat{\tau} + \frac{\partial^2\eta}{\partial s^2} + \widehat{\tau}\frac{\partial\gamma}{\partial s} + \frac{\partial^2\eta}{\partial s^2} + \frac{\partial\widehat{\kappa}}{\partial s} + \lambda\widehat{\tau} + \frac{\partial\eta}{\partial s}\widehat{\kappa} + \eta\widehat{\kappa}^2)\widehat{B},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\widehat{T}}{\partial s} =& \frac{\partial}{\partial t}(\widehat{\kappa}\widehat{T} + \widehat{N}) = (\frac{\partial\widehat{\kappa}}{\partial t} - (\lambda + \frac{\partial\gamma}{\partial s} + \gamma\widehat{\kappa}))\widehat{T} + (\lambda\widehat{\kappa} + \widehat{\kappa}\frac{\partial\gamma}{\partial s} + \gamma\widehat{\kappa}^2)\widehat{N}\\ & + (\gamma\widehat{\kappa}\widehat{\tau} + \widehat{\kappa}\frac{\partial\eta}{\partial s} + \eta\widehat{\kappa}^2 + \Omega)\widehat{B}.\end{aligned}$$

Hence we have

$$\frac{\partial}{\partial s}\frac{\partial\widehat{T}}{\partial t} = \frac{\partial}{\partial s}\frac{\partial\widehat{T}}{\partial t}, \tag{16}$$

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial\widehat{B}}{\partial t} =& -(\frac{\partial}{\partial s}(\gamma\widehat{\tau}) + \widehat{\kappa}\gamma\widehat{\tau} + \frac{\partial^2\eta}{\partial s^2} + \widehat{\kappa}\frac{\partial\eta}{\partial s} + \frac{\partial}{\partial s}(\eta\widehat{\kappa}) + \eta\widehat{\kappa}^2)\widehat{T}\\ & - (\gamma\widehat{\tau} + \frac{\partial\eta}{\partial s} + \frac{\partial\Omega}{\partial s} + \Omega\widehat{\kappa} + \eta\widehat{\kappa})\widehat{N} - \Omega\widehat{\tau}\widehat{B},\end{aligned} \tag{17}$$

and

$$\frac{\partial}{\partial t}\frac{\partial\widehat{B}}{\partial s} = \frac{\partial}{\partial t}(\widehat{\kappa}\widehat{B}) = \frac{\partial\widehat{\kappa}}{\partial t}\widehat{B} - (\gamma\widehat{\kappa}\widehat{\tau} + \widehat{\kappa}\frac{\partial\eta}{\partial s} + \eta\widehat{\kappa}^2)\widehat{T} - \Omega\widehat{\kappa}\widehat{N}. \tag{18}$$

We get

$$\frac{\partial}{\partial s}\frac{\partial\widehat{B}}{\partial t} = \frac{\partial}{\partial s}\frac{\partial\widehat{B}}{\partial t}. \tag{19}$$

Using (17) and (18) in (19), we have evolution of equiform curvature in double isotropic space

$$\frac{\partial\widehat{\kappa}}{\partial t} = -\Omega\widehat{\tau} = -(\frac{\partial^2\eta}{\partial s^2} + 3\widehat{\tau}\frac{\partial\gamma}{\partial s} + \frac{\partial\widehat{\kappa}}{\partial s} + \widehat{\tau}\lambda + \widehat{\kappa}\lambda\widehat{\tau})\widehat{\tau}.$$

Using (13)

$$\frac{\partial}{\partial s}\frac{\partial\widehat{N}}{\partial t} = (-(\lambda\widehat{\kappa} + \widehat{\kappa}\frac{\partial\gamma}{\partial s} + \widehat{\kappa}^2\gamma) - \frac{\partial}{\partial s}(\lambda + \frac{\partial\gamma}{\partial s} + \gamma\widehat{\kappa}))\widehat{N} + (\frac{\partial\Omega}{\partial s} + \Omega\widehat{\kappa})\widehat{B} \tag{20}$$

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\widehat{N}}{\partial s} =& \frac{\partial}{\partial t}(\widehat{\kappa}\widehat{N} + \widehat{\tau}\widehat{B})\\ =& (\frac{\partial\widehat{\kappa}}{\partial t} - \Omega\widehat{\tau})\widehat{N} - (\widehat{\kappa}\lambda + \widehat{\kappa}\frac{\partial\gamma}{\partial s} + \eta\widehat{\kappa}^2 + \eta\widehat{\tau}^2 + \widehat{\tau}\frac{\partial\eta}{\partial s} + \eta\widehat{\kappa}\widehat{\tau})\widehat{T} + (\frac{\partial\widehat{\tau}}{\partial t} + \Omega\widehat{\kappa})\widehat{B}.\end{aligned} \tag{21}$$

From

$$\frac{\partial}{\partial s}\frac{\partial \widehat{N}}{\partial t}=\frac{\partial}{\partial s}\frac{\partial \widehat{N}}{\partial t} \tag{22}$$

(21) and (22), it can be derived

$$\frac{\partial \widehat{\tau}}{\partial t}=\frac{\partial \Omega}{\partial s}.$$

4 Conclusion

Double isotropic space have important application fields like soliton theory. In this work, we present that double inextensible flows in the equiform geometry of double isotropic space $\mathscr{I}_3^{(2)}$. We obtain necessary and sufficient conditions for an double inextensible curve flow. We give characterizations for evolution of first equiform curvature and second equiform curvature in the equiform geometry of double isotropic space.

References

[1] Kwon D. Y, Park F.C, Evolution of inelastic plane curves, Applied Mathematics Letters 12 (1999) 115-119.

[2] Kwon D Y, Park F C, Chi, D P, Inextensible flows of curves and developable surfaces, Applied Mathematics Letters 18 (2005) 1156-1162.

[3] Mochiyama H, E. Shimemura E, Kobayashi H, Shape control of manipulators with hyperdegrees of freedom, Int. J. Robot. Res.18 (1999) 584-600.

[4] Kass M, Witkin A, Terzopoulos D, Snakes: active contour models: 1 st Int. Conference on computer vision (1987) 259-268.

[5] Chirikjian G , Burdick J, Kinematics of Hyper-Redundant manipulations, In Proc. Asme Mechanisms Conference (1990) 391-396.

[6] Gage M, On area preserving evolution equation for plane curves, Contemporary Mathematics 51 (1986) 51-62.

[7] Gage M, Hamilton R, The heat equation shrinking convex plane curves, J Differential Geometry 23 (1986) 69-96.

[8] Grayson, M, The heat equation shrinks embedded plane curves to round points, J.Differential Geometry 26 (1987) 285-314.

[9] Gurbuz N, Inextensible Flows of Spacelike Timelike and Null Curves, Int.J.Contemp. Math. Sciences 4 (2009) 1599-1604.

[10] Bektas M, Kulahci M. Inextensible Flows of Space-like Curves in Lightlike Cone, Prespacetime Journal 6 (2015) 313-321.

[11] Yildiz O G, Tosun M, Karakus S, A note on inextensible flows of curves in E^n, International Electronic Journal of Geometry 6 (2013) 118-124.

[12] Yizdiz O G, Tosun M, A Note on inextensible Flows of Curves in E_1^n. $arXiv$:1302.6082v1 (math.DG) 25 Feb 2013.

[13] Gurbuz N, Inextensible flows of non-null curves on an pseudo-Euclidean hypersuface in pseudo-Euclidean space R_1^n, Life Science Journal 10 (2013) 1084.

[14] Yoon D. W, Inelastic flows of curves according to equiform in Galilean space, Journal of the Chungheong Mathematical Society 24 (2011) 665-673.

[15] Pavkovic B J, Kamenarovic I, The general solution of the Frenet's System in the double isotropic space $\mathscr{I}_3^{(2)}$, Rad Jazu 428 (1987) 17-24.

[16] Pavkovic B.J, Equiform geometry of curves in the isotropic spaces $\mathscr{I}_3^{(1)}$ and $\mathscr{I}_3^{(2)}$,Rad JAZU (1986) 39-44.

[17] Erjavec Z, Dvjak B, Horvat D, The general solutions of Frenet's System in the equiform geometry of the Galilean, Pseudo-Galilean, simple isotropic and double isotropic space, International Mathematical Forum 6 (2011) 837-856.