New Trends in Mathematical Sciences
http://www.ntmsci.com

Some new traveling wave solutions of the modified Benjamin-Bona-Mahony equation via the improved (G'/G)-expansion method

Hasibun Naher [1,2], *Farah Aini Abdullah* [2] *and Ahmet Bekir* [3]

[1] BRAC University, Department of Mathematics and Natural Sciences, Mohakhali, Dhaka 1212, Bangladesh.
[2] School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Penang, Malaysia.
[3] Eskisehir Osmangazi University, Department of Mathematics-Computer,26480 Eskisehir, Turkey.

Received: 16 December 2014, Revised: 26 December 2014, Accepted: 27 December 2014
Published online: 31 December 2014

Abstract: The improved (G'/G)-expansion method is a powerful mathematical tool for solving nonlinear evolution equations which arise in mathematical physics, engineering sciences and other technical arena. In this article, we construct some new exact traveling wave solutions for the modified Benjamin-Bona-Mahony equation by applying the improved (G'/G)-expansion method. In the method, the general solution of the second order linear ordinary differential equation with constant coefficients is used for studying nonlinear partial differential equations. The solution procedure of this method is executed by algebraic software, such as, Maple. The obtained solutions including solitary and periodic wave solutions are presented in terms of the hyperbolic function, the trigonometric function and the rational forms. It is noteworthy to reveal that some of our solutions are in good agreement with the published results for special cases which certifies our other solutions. Furthermore, the graphical presentations of some solutions are illustrated in the figures.
Keywords: Traveling wave solutions, the modified Benjamin-Bona-Mahony equation, the improved (G'/G)-expansion method, nonlinear evolution equations.

Keywords: Benjamin-Bona-Mahony equation, traveling wave solution, (G'/G)-expansion method.

1 Introduction

Partial differential equations (PDEs) have been a most important subject of study in all areas of mathematical physics, engineering sciences and other technical arena. At present time, different methods are being established to construct traveling wave solutions of nonlinear PDEs, such as, the tanh method [1], the homogeneous balance method [2], the variational iteration method [3]-[7], the Backlund transformation method [8], the Hirota bilinear transformation method [9], the Jacobi elliptic function expansion method [10,11], the Cole-Hopf transformation method [12], the direct algebraic method [13], the Exp-function method [14]-[19], the Lie symmetry method [20,21], the homotopy analysis method [22] and others [23]-[26].

Recently, Wang et al. [27] presented a method called the (G'/G)-expansion method and obtained exact traveling wave solutions of nonlinear PDEs. In this method, $u(\xi) = \sum_{i=0}^{m} a_i(\frac{G'}{G})^i$, is used, as the traveling wave solutions, where $a_m \neq 0$. After that, many researchers executed this method to establish analytical solutions of the nonlinear PDEs [28]-[41].

More lately, Zhang et al. [42] expanded the basic (G'/G)-expansion method, and called the improved (G'/G)-expansion method for constructing traveling wave solutions of some nonlinear PDEs. In the method, they used,

* Corresponding author e-mail: hasibun06tasauf@gmail.com,farahaini@usm.my and abekir@ogu.edu.tr

$u(\xi)=\sum_{j=-m}^{m} a_j(\frac{G'}{G})^j$, as the traveling wave solutions, where a_{-m} or a_m may be zero, but both a_{-m} and a_m cannot be zero at a time.

Afterwards, many researchers implemented this powerful and straightforward method to obtain many new traveling wave solutions of various nonlinear PDEs. For instance, Zhao et al. [43] studied the variant Boussinesq equations via this method to construct traveling wave solutions while Hamad et al. [44] concerned about the same method to obtain analytical solutions for the higher dimensional potential YTSF equation. In Ref. [45] Naher and Abdullah studied nonlinear reaction diffusion equation to construct traveling wave solutions via this method whilst they [46] obtained abundant analytical solutions of the (2+1)-dimensional modified Zakharov-Kuznetsov equation by applying the same method. Nofel et al. [47] investigated the fifth order KdV equation by using this method to find exact wave solutions and so on.

Many researchers studied the modified Benjamin-Bona-Mahony equation for obtaining exact traveling wave solutions by using different methods. For example, Yusufoglu and Bekir [48] investigated this equation by applying the tanh and sine-cosine methods to find exact solutions while Yusufoglu [49] implemented the Exp-function method to construct traveling wave solutions of the same equation. In Ref. [50], Taghizadeh and Mirzazadeh used modified extended tanh method to establish traveling wave solutions of this equation. Abbasbandy and Shirzadi [51] executed the first integral method to seek analytical solutions of the same equation. Aslan [52] studied the same equation to find exact solutions by utilizing the basic (G'/G)-expansion method. But, to the best of our knowledge, the modified Benjamin-Bona-Mahony equation is not investigated via the improved (G'/G)-expansion method to construct traveling wave solutions. The main difference between the improved (G'/G)-expansion method and the basic-expansion method is that the traveling wave solutions $u(\xi)=\sum_{j=-m}^{m} a_j(\frac{G'}{G})^j$ is used instead of $u(\xi)=\sum_{i=0}^{m} a_i(\frac{G'}{G})^i$.

In this article, we study the partial differential equation, namely, the modified Benjamin-Bona-Mahony equation via the improved (G'/G)-expansion to obtain abundant new exact traveling wave solutions including solitons and periodic solutions.

2 Explanation of the improved -expansion method for the nonlinear PDEs

Suppose the general nonlinear partial differential equation:

$$R(w,w_t,w_x,w_{xx},w_{xxx},w_{xt},...)=0, \tag{1}$$

where $w=w(x,t)$ is an unknown function, R is a polynomial in $w=w(x,t)$ and the subscripts indicate the partial derivatives.

The main steps of the improved (G'/G)-expansion method [42] are:

Step 1. Consider the traveling wave variable:

$$w(x,t)=p(\theta), \quad \theta=Kx+Ht, \tag{2}$$

Now using Eq. (2), Eq. (1) is converted into an ordinary differential equation (ODE) for $p(\theta)$:

$$S(p,p',p'',p''',...)=0, \tag{3}$$

where the superscripts stand for the ordinary derivatives with respect to θ.

Step 2. According to possibility, Eq. (3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero, for simplicity.

Step 3. Suppose that the traveling wave solution of Eq. (3) can be expressed in the form [42]:

$$p(\theta)=\sum_{j=-n}^{n} r_j(\frac{G'}{G})^j \tag{4}$$

with $G=G(\theta)$ satisfies the second order linear ODE:

$$G''+\lambda G'+\mu G=0, \tag{5}$$

where $r_j(j=0,\pm1,\pm2,...,\pm n),\lambda$ and μ are constants, where either r_{-n} or r_n may be zero, but both r_{-n} and r_n cannot be zero at a time.

Step 4. To determine the value of n, substituting Eq. (4) along with Eq. (5) into Eq. (3) and then taking the homogeneous balance between the highest order derivatives and the highest order nonlinear terms appearing in Eq. (3).

Step 5. Substitute Eq. (4) and Eq. (5) into Eq. (3) with the value of obtained in Step 4. Equating the coefficients of $(\frac{G'}{G})^k,(k=0,\pm1,\pm2,...)$ then setting each coefficient to zero, we obtain a set of algebraic equations for $r_j(j=0,\pm1,\pm2,...,\pm n),H,\lambda$ and μ.

Step 6. Solve the system of algebraic equations obtained in step 5 by the help of algebraic software Maple and yields values for $r_j(j=0,\pm1,\pm2,...,\pm n),H,\lambda$ and μ. Then, substituting obtained values in Eq. (4) along with Eq. (5) with the value of n, we obtain traveling wave solutions of the partial differential Eq. (1).

3 Applications of the method

In this section, we study the modified Benjamin-Bona-Mahony equation by applying the improved (G'/G)-expansion method to construct exact traveling wave solutions.

3.1 The modified Benjamin-Bona-Mahony equation

In this work, we consider the modified Benjamin-Bona-Mahony equation followed by

$$u_t+\alpha u_x+\beta u^2u_x-\gamma u_{xxt}=0. \tag{6}$$

where λ,γ are free parameters and $\beta\neq 0$.
Now, we use the wave transformation Eq. (2) into Eq. (6), which yields:

$$(H+K\alpha)p'+K\beta p^2p'-HK^2\gamma p'''=0, \tag{7}$$

where the superscripts stand for the derivatives with respect to θ.

Eq. (7) is integrable, therefore, integrating with respect to once yields:

$$(H+K\alpha)p+\frac{K\beta}{3}p^3-HK^2\gamma p''+C=0, \tag{8}$$

where C is an integral constant which is to be determined later.

Taking the homogeneous balance between p^3 and p'' in Eq. (8), we obtain $n = 1$. So, the solution of Eq. (8) is:

$$p(\theta) = r_{-1}(\frac{G'}{G})^{-1} + r_0 + r_1(\frac{G'}{G}) \tag{9}$$

where r_{-1}, r_0 and $r_1$are constants.

Substituting Eq. (9) together with Eq. (5) into the Eq. (8), the left-hand side of Eq. (8) is converted into a polynomial of $(\frac{G'}{G})^k, (k = 0, \pm 1, \pm 2, ...)$. Collecting all terms with the same power of $(\frac{G'}{G})$ according to step 5. Then setting each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for $r_{-1}, r_0, r_1, H, C, \lambda$ and μ.(which are not shown, for simplicity).

Solving the system of obtained algebraic equations with the aid of algebraic software Maple, we obtain three different values.

Case 1:

$$\begin{gathered} r_{-1} = 0, \qquad r_0 = \pm K\lambda\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}, \\ r_1 = \pm 2K\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}, \qquad H = \frac{-2\alpha K}{2+K^2\gamma(\lambda^2-4\mu)}, C = 0 \end{gathered} \tag{10}$$

where λ, γ are free parameters and $\beta \neq 0$.

Case 2:

$$\begin{gathered} r_{-1} = \pm 2K\mu\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}, \quad r_0 = \pm K\lambda\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}, \\ r_1 = 0, \qquad H = \frac{-2\alpha K}{2+K^2\gamma(\lambda^2-4\mu)}, C = 0, \end{gathered} \tag{11}$$

where λ, γ are free parameters and $\beta \neq 0$.

Case 3:

$$\begin{gathered} r_{-1} = \pm 2K\mu\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2+8\mu))}}, \qquad r_0 = \pm K\lambda\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2+8\mu))}}, \\ r_1 = \pm 2K\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2+8\mu))}}, \qquad H = \frac{-2\alpha K}{2+K^2\gamma(\lambda^2+8\mu)}, \\ C = \pm\frac{8K^4\alpha\gamma\mu\lambda}{2+K^2\gamma(\lambda^2+8\mu)}\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2+8\mu))}} \end{gathered} \tag{12}$$

where λ, γ are free parameters and $\beta \neq 0$.

We obtain three different families of traveling wave solutions of Eq. (8) by substituting the general solution Eq. (5) into Eq. (9):

Family 1: Hyperbolic function solutions:

When $\lambda^2 - 4\mu > 0$, we obtain

$$p(\theta)=r_{-1}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{X\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)+Y\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)}{X\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)+Y\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)}\right)-\frac{\lambda}{2}\right)^{-1}$$
$$+r_0+r_1\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{X\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)+Y\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)}{X\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)+Y\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)}\right)-\frac{\lambda}{2}\right) \tag{13}$$

where X and Y are arbitrary constants. If X and Y are taken particular values, various known solutions can be rediscovered.

Family 2: Trigonometric function solutions:
When $\lambda^2-4\mu<0$, we obtain

$$p(\theta)=r_{-1}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{-X\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)+Y\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)}{X\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)+Y\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)}\right)-\frac{\lambda}{2}\right)^{-1}$$
$$+r_0+r_1\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{-X\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)+Y\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)}{X\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)+Y\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta)}\right)-\frac{\lambda}{2}\right) \tag{14}$$

where X and Y are arbitrary constants. If X and Y are taken particular values, various known solutions can be rediscovered.

Family 3: Rational function solutions:
When $\lambda^2-4\mu=0$, we obtain

$$p(\theta)=r_{-1}\left(\frac{Y}{X+Y\theta}-\frac{\lambda}{2}\right)^{-1}+r_0+r_1\left(\frac{Y}{X+Y\theta}-\frac{\lambda}{2}\right). \tag{15}$$

Family 1: Hyperbolic function solutions:
Now, substituting Eqs. (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (9), yields the hyperbolic function solution Eq. (13), we obtain following traveling wave solutions respectively (if $X=0$ but $Y\neq 0$):

$$p_1(x,t)=\pm K\sqrt{\frac{-3\alpha\gamma(\lambda^2-4\mu)}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}\coth\frac{\sqrt{\lambda^2-4\mu}}{2}\theta,$$

$$p_2(x,t)=\pm K\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}\left(2\mu\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\coth\frac{\sqrt{\lambda^2-4\mu}}{2}\theta-\frac{\lambda}{2}\right)^{-1}+\lambda\right),$$

where $\theta=Kx-\frac{2\alpha K}{2+K^2\gamma(\lambda^2-4\mu)}t$.

$$p_3(x,t)=\pm K\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}\left(\begin{array}{c}2\mu\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\coth\frac{\sqrt{\lambda^2-4\mu}}{2}\theta-\frac{\lambda}{2}\right)^{-1}\\+\sqrt{\lambda^2-4\mu}\coth\frac{\sqrt{\lambda^2-4\mu}}{2}\theta\end{array}\right),$$

where $\theta=Kx-\frac{2\alpha K}{2+K^2\gamma(\lambda^2+8\mu)}t$.

Also, substituting Eqs. (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (9), yields the hyperbolic function solution Eq. (13), our traveling wave solutions become respectively (if $X = 0$ but $Y \neq 0$):

$$p_4(x,t) = \pm K\sqrt{\frac{-3\alpha\gamma(\lambda^2-4\mu)}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}\tanh\frac{\sqrt{\lambda^2-4\mu}}{2}\theta,$$

$$p_5(x,t) = \pm K\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}\left(2\mu\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\frac{\sqrt{\lambda^2-4\mu}}{2}\theta-\frac{\lambda}{2}\right)^{-1}+\lambda\right),$$

$$p_6(x,t) = \pm K\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}\left(\begin{array}{c}2\mu\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\frac{\sqrt{\lambda^2-4\mu}}{2}\theta-\frac{\lambda}{2}\right)^{-1}\\ +\sqrt{\lambda^2-4\mu}\tanh\frac{\sqrt{\lambda^2-4\mu}}{2}\theta\end{array}\right),$$

Again, substituting Eqs. (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (9), yields the hyperbolic function solution Eq. (13), our obtained exact solutions become respectively. (if $X \neq 0$ but $X > Y$):

$$p_7(x,t) = \pm K\sqrt{\frac{-3\alpha\gamma(\lambda^2-4\mu)}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta+\theta_0\right),$$

$$p_8(x,t) = \pm K\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}\left(2\mu\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta+\theta_0\right)-\frac{\lambda}{2}\right)^{-1}+\lambda\right),$$

where $\theta_0 = \tanh^{-1}\frac{Y}{X}$ and $\theta = Kx - \frac{2\alpha K}{2+K^2\gamma(\lambda^2-4\mu)}t$.

$$p_9(x,t) = \pm K\sqrt{\frac{-3\alpha\gamma}{\beta(2+K^2\gamma(\lambda^2-4\mu))}}\left(\begin{array}{c}2\mu\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta+\theta_0\right)-\frac{\lambda}{2}\right)^{-1}\\ +\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\theta+\theta_0\right)\end{array}\right),$$

where $\theta_0 = \tanh^{-1}\frac{Y}{X}$ and $\theta = Kx - \frac{2\alpha K}{2+K^2\gamma(\lambda^2+8\mu)}t$.

Family 2: Trigonometric function solutions:

Substituting Eqs. (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (9), yields the trigonometric function solution Eq. (14), our exact traveling wave solutions become respectively (if $X = 0$ but $Y \neq 0$):

$$p_{10}(x,t) = \pm K\sqrt{\frac{3\alpha\gamma(4\mu-\lambda^2)}{\beta(2+K^2\gamma(4\mu-\lambda^2))}}\cot\frac{\sqrt{4\mu-\lambda^2}}{2}\theta,$$

$$p_{11}(x,t) = \pm K\sqrt{\frac{3\alpha\gamma}{\beta(2+K^2\gamma(4\mu-\lambda^2))}}\left(2\mu\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\cot\frac{\sqrt{4\mu-\lambda^2}}{2}\theta-\frac{\lambda}{2}\right)^{-1}+\lambda\right),$$

where $\theta = Kx - \frac{2\alpha K}{2+K^2\gamma(\lambda^2-4\mu)}t$.

$$p_{12}(x,t) = \pm K\sqrt{\frac{3\alpha\gamma}{\beta(2+K^2\gamma(4\mu-\lambda^2))}}\left(\begin{array}{c}2\mu\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\cot\frac{\sqrt{4\mu-\lambda^2}}{2}\theta-\frac{\lambda}{2}\right)^{-1}\\ +\sqrt{\lambda^2-4\mu}\cot\frac{\sqrt{4\mu-\lambda^2}}{2}\theta\end{array}\right),$$

where $\theta = Kx - \frac{2\alpha K}{2+K^2\gamma(\lambda^2+8\mu)}t$.

Furthermore, substituting Eqs. (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (9), yields the trigonometric function solution Eq. (14), our obtained solutions become respectively (if $Y=0$ but $X\neq 0$)

$$p_{13}(x,t)=\pm K\sqrt{\frac{3\alpha\gamma(4\mu-\lambda^2)}{\beta(2+K^2\gamma(4\mu-\lambda^2))}}\tan\frac{\sqrt{4\mu-\lambda^2}}{2}\theta,$$

$$p_{14}(x,t)=\pm K\sqrt{\frac{3\alpha\gamma}{\beta(2+K^2\gamma(4\mu-\lambda^2))}}\left(2\mu\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\frac{\sqrt{4\mu-\lambda^2}}{2}\theta-\frac{\lambda}{2}\right)^{-1}+\lambda\right),$$

$$p_{15}(x,t)=\pm K\sqrt{\frac{3\alpha\gamma}{\beta(2+K^2\gamma(4\mu-\lambda^2))}}\left(\begin{array}{c}2\mu\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\frac{\sqrt{4\mu-\lambda^2}}{2}\theta-\frac{\lambda}{2}\right)^{-1}\\ +\sqrt{\lambda^2-4\mu}\tan\frac{\sqrt{4\mu-\lambda^2}}{2}\theta\end{array}\right),$$

Moreover, substituting Eqs.(10), (11) and (12) together with the general solution Eq. (5) into the Eq. (9), yields the trigonometric function solution Eq. (14), we obtain following solutions respectively (if $X\neq 0$ but $X>Y$):

$$p_{16}(x,t)=\pm K\sqrt{\frac{3\alpha\gamma(4\mu-\lambda^2)}{\beta(2+K^2\gamma(4\mu-\lambda^2))}}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\theta-\theta_0\right),$$

$$p_{17}(x,t)=\pm K\sqrt{\frac{3\alpha\gamma}{\beta(2+K^2\gamma(4\mu-\lambda^2))}}\left(2\mu\left(\frac{\sqrt{4\mu-\lambda^2\mu}}{2}\tan\left(\frac{\sqrt{4\mu-\lambda^2\mu}}{2}\theta-\theta_0\right)-\frac{\lambda}{2}\right)^{-1}+\lambda\right),$$

$$p_{18}(x,t)=\pm K\sqrt{\frac{3\alpha\gamma}{\beta(2+K^2\gamma(4\mu-\lambda^2\mu))}}\left(\begin{array}{c}2\mu\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\tanh\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\theta-\theta_0\right)-\frac{\lambda}{2}\right)^{-1}\\ +\sqrt{4\mu-\lambda^2}\tanh\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\theta-\theta_0\right)\end{array}\right),$$

Family 3: Rational function solutions:

Substituting Eqs. (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (9), we obtain rational function solution Eq. (14), and our obtained exact solutions become respectively (if $\lambda^2-4\mu=0$):

$$p_{19}(x,t)=\pm K\sqrt{\frac{-3\alpha\gamma}{2\beta}}+\frac{2Y}{X+Y\theta},$$

$$p_{20}(x,t)=\pm K\sqrt{\frac{-3\alpha\gamma}{2\beta}}\left(2\mu\left(\frac{Y}{X+Y\theta}-\frac{\lambda}{2}\right)^{-1}+\lambda\right),$$

where $\theta=Kx-\frac{2\alpha K}{2+K^2\gamma(\lambda^2-4\mu)}t$.

$$p_{21}(x,t)=\pm K\sqrt{\frac{-3\alpha\gamma}{2\beta}}\left(2\mu\left(\frac{Y}{X+Y\theta}-\frac{\lambda}{2}\right)^{-1}+\frac{2Y}{X+Y\theta}\right),$$

where $\theta=Kx-\frac{2\alpha K}{2+K^2\gamma(\lambda^2+8\mu)}t$.

4 Results and discussion

It is important to state that some of our solutions are in good agreement with the published results and some are new in this article. In addition, the graphical presentations of some obtained solutions are described in the following subsection in figure 1 to figure 12.

4.1 Table. Comparison between Aslan [52] solutions and our solutions:

Aslan [52] solutions	Our solutions
i. If $C_1 \neq 0, C_2=0, \lambda=3, \mu=2$, $\alpha=\beta=\gamma=1$and $k=1$, solution (33) (from: subsection 3.3) becomes: $u_{1,2}(x,t) = \pm i \coth \frac{1}{2}(x-\frac{2}{3}t)$.	i. If $\lambda=3, \mu=2, \alpha=\beta=\gamma=1$, $K=1$ and $p_1 = u_{1,2}$ solution $p_1(\theta)$ becomes: $u_{1,2}(x,t) = \pm i \coth \frac{1}{2}(x-\frac{2}{3}t)$.
ii. If $C_1=0, C_2 \neq 0, \lambda=3, \mu=2$, $\alpha=\beta=\gamma=1$and $k=1$, solution (33) (from: subsection 3.3) becomes: $u_{1,2}(x,t) = \pm i \tanh \frac{1}{2}(x-\frac{2}{3}t)$.	ii. If $\lambda=3, \mu=2, \alpha=\beta=\gamma=1$, $K=1$ and $p_4 = u_{1,2}$ solution $p_1(\theta)$ becomes: $u_{1,2}(x,t) = \pm i \tanh \frac{1}{2}(x-\frac{2}{3}t)$.
iii. If $C_1=0, C_2 \neq 0, \lambda=4, \mu=5$, $\alpha=\beta=\gamma=1$and $k=1$, solution (34) (from: subsection 3.3) becomes: $u_{3,4}(x,t) = \pm\sqrt{2}\cot i(x-\frac{1}{3}t)$.	iii. If $\lambda=4, \mu=5, \alpha=\beta=\gamma=1$, $K=1$ and $p_{10} = u_{3,4}$ solution $p_{10}(\theta)$ becomes: $u_{3,4}(x,t) = \pm\sqrt{2}\cot i(x-\frac{1}{3}t)$.
iv. If $C_1 \neq 0, C_2=0, \lambda=4, \mu=5$, $\alpha=\beta=\gamma=1$and $k=1$, solution (34) (from: subsection 3.3) becomes: $u_{3,4}(x,t) = \pm\sqrt{2}\tan i(x-\frac{1}{3}t)$.	iv. If $\lambda=4, \mu=5, \alpha=\beta=\gamma=1$, $K=1$ and $p_{13} = u_{3,4}$ solution $p_{13}(\theta)$ becomes: $u_{3,4}(x,t) = \pm\sqrt{2}\tan i(x-\frac{1}{3}t)$.
v. If $C_3=1, C_2=0, \lambda^2-4\mu=0$, $\alpha=\beta=\gamma=1$and $k=1$, solution (35) (from: subsection 3.3) becomes: $u_{5,6}(x,t) = \pm\frac{1}{x-t})$.	v. If $X=0, Y=1, \lambda^2\text{-}4\mu=0, \alpha=\gamma=1$, $\beta=6, K=1$ and $p_{19} = u_{5,6}$ solution $p_{19}(\theta)$ becomes: $u_{5,6}(x,t) = \pm\frac{1}{x-t})$.
vi. If $\lambda=3, \mu=2, \alpha=3, \beta=\text{-}3, \gamma=1$, $\xi_0 = 0$ and $k=1$, solution (36) (from: subsection 3.3) becomes: $u_{7,8}(x,t) = \pm\tanh\frac{1}{2}(x-2t)$.	vi. If $\lambda=3, \mu=2, \alpha=3, \beta=\text{-}3, \gamma=1$, $K=1, \theta_0=0$ and $p_7 = u_{7,8}$ solution $p_7(\theta)$ becomes: $u_{7,8}(x,t) = \pm\tanh\frac{1}{2}(x-2t)$.

Beside this table, we obtain new traveling wave solutions $p_2, p_3, p_5, p_6, p_8, p_9, p_{11}, p_{12}, p_{14}$ to p_{18}, p_{20} and p_{21} which have not been found in the previous literature.

4.2 Graphical representations of the solutions

Some of our solutions are shown in the figures by using the commercial software Maple:

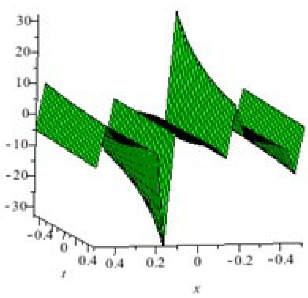

Fig. 1: Periodic solution for

$\lambda=4, \mu=6, \alpha=2, \beta=3, \gamma=4, K=7$

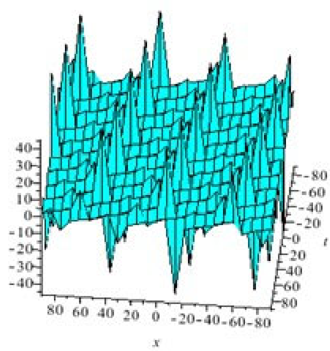

Fig. 2: Soliton solution for

$\lambda=4, \mu=5, \alpha=2, \beta=2, \gamma=1, K=3$

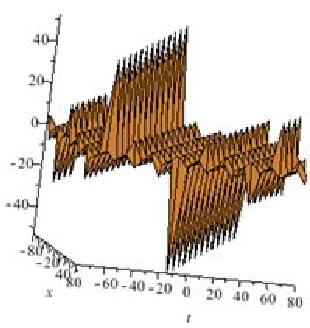

Fig. 3: Solitons solution for

$\lambda = 5, \mu = 7, \alpha = 3, \beta = 3, \gamma = 2, K = 5$

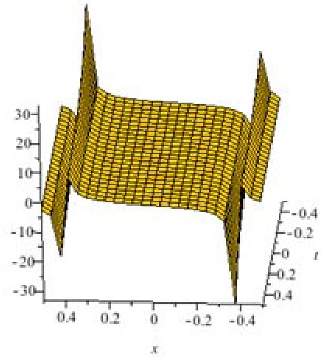

Fig. 4: Periodic solution for

$\lambda = 6, \mu = 10, \alpha = 1, \beta = 3, \gamma = 3, K = 4$

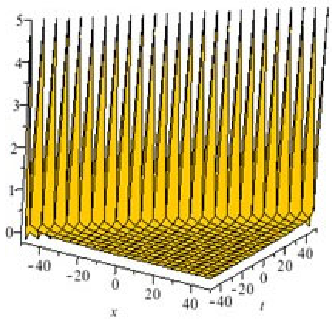

Fig. 5: Solitons solution for

$\lambda = 4, \mu = 4, \alpha = 1, \beta = 6, \gamma = 1, K = 5, X = 1, Y = 1$

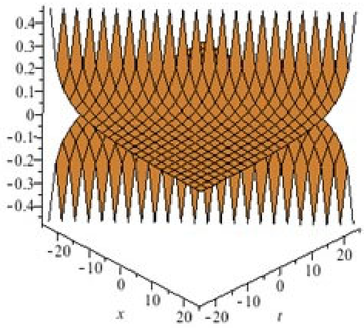

Fig. 6: Solitons solution for

$\lambda = 6, \mu = 9, \alpha = 1, \beta = 6, \gamma = 1, K = 1, X = 0, Y = 1$

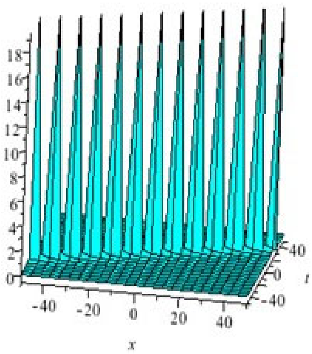

Fig. 7: Solitons solution for

$\lambda = 4, \mu = 4, \alpha = 2, \beta = 6, \gamma = 3, K = 8, X = 1, Y = 1$

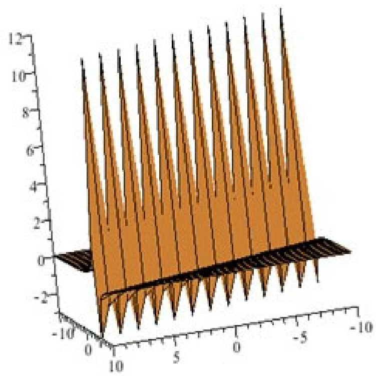

Fig. 8: Solitons solution for

$\lambda = 4, \mu = 4, \alpha = 2, \beta = 3, \gamma = 1, K = 1, X = 1, Y = 1$

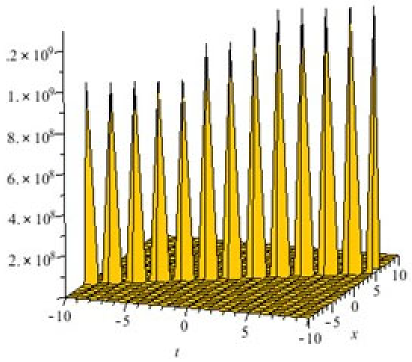

Fig. 9: Solitons solution for

$\lambda = 4, \mu = 3, \alpha = 3, \beta = 2, \gamma = 5, K = 1$

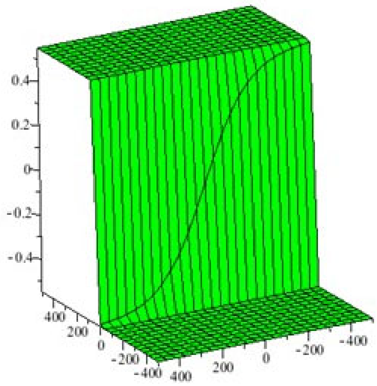

Fig. 10: Periodic solution for

$\lambda = 6, \mu = 6, \alpha = 0.5, \beta = 5, \gamma = 7, K = 15$

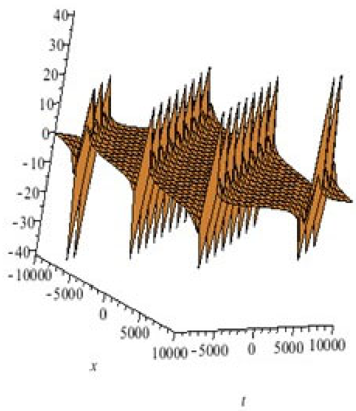

Fig. 11: Solitons solution for

$\lambda = 3, \mu = 4, \alpha = 1, \beta = 1, \gamma = 1, K = 1$

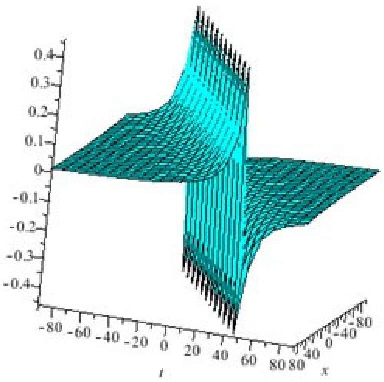

Fig. 12: Solitons solution for

$\lambda = 8, \mu = 16, \alpha = 2, \beta = 0.5, \gamma = 0.5, K = 0.5, X = 0, Y = 1$

5 Conlusion

In this article, we obtain abundant exact traveling wave solutions of the modified Benjamin-Bona-Mahony equation via the improved (G'/G)-expansion method. The obtained solutions including solitons and periodic solutions are presented in terms of the hyperbolic, the trigonometric and the rational forms. Further, it is important mentioning that some of our solutions are in good agreement with the published results and some are new which validates our other solutions. Moreover, the solutions show that the used method is more effective and powerful mathematical tool and gives many new solutions. We expect that nonlinear PDEs which are arising in mathematical physics, engineering sciences and other technical arena will be investigated by applying this straightforward method.

References

[1] Malfliet, W., Solitary wave solutions of nonlinear wave equations,Am. J. Physics, 60, 650-654 (1992)

[2] Wang, M.L., Zhou, Y.B., Li, Z.B., Application of homogeneous balance method to exact solutions of nonlinear equations in mathematical physics Phys. Let. A, 216, 67-75 (1996)

[3] He, J. ,Variational iteration method for delay differential equations, Communication in Nonlinear Science and Numerical Simulation, 2(4), 235-236 (1997)
[4] Abdou, M.A., Soliman, A.A., Variational iteration method for solving Burger's and coupled Burger's equations, J. Comput. Appl. Math. 181, 245-251 (2005)
[5] Yusufoğlu, E., Bekir, A., The variational iteration method for solitary patterns solutions of gBBM equation, Physics Letter A, 367 (6) 461-464 (2007)
[6] Younesian, D., Askari, H., Saadatnia, Z., Yildirim, A., Analytical solution for nonlinear wave propagation in shallow media using the variational iteration method, Waves in Random and Complex Media, 22 (2), 133-142 (2012)
[7] Abbasbandy, S., Numerical solution of non-linear Klein-Gordon equations by variational iteration method, Int. J. Numer. Meth. Engng 70, 876-881 (2007)
[8] Rogers, C., Shadwick, W.F., Backlund Transformations, Academic Press, New York. (1982)
[9] Hirota, R., Exact solution of the KdV equation for multiple collisions of solitons, Phys. Rev. Lett. 27, 1192-1194 (1971)
[10] Liu, S., Fu, Z., Liu, S., Zhao, Q., Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A, 289, 69–74 (2001)
[11] Ebadi, G., Yousefzadeh, N., Triki, H., Yildirim, A., & Biswas, A., Envelope solitons, periodic waves and other solutions to Boussinesq-Burgers equation, Romanian Reports in Physics, 64(4), 915-932 (2012)
[12] Salas, A.H., Gomez, C.A., Application of the Cole-Hopf transformation for finding exact solutions to several forms of the seventh-order KdV equation, Mathematical Problems in Engineering, 14, 194329 (2010)
[13] Soliman, A.A. Abdo, H.A., New exact Solutions of nonlinear variants of the RLW, the PHI-four and Boussinesq equations based on modified extended direct algebraic method, International Journal of Nonlinear Science, 7, 3 274-282 (2009)
[14] He, J.H., Wu, X.H., Exp-function method for nonlinear wave equations, Chaos Solitons and Fractals, 30 700-708 (2006)
[15] Bekir, A., Boz, A., Exact solutions for nonlinear evolution equations using Exp-function method, Phys. Letters A, 372 1619-1625 (2008)
[16] Naher, H., Abdullah, F.A., Akbar, M.A., The exp-function method for new exact solutions of the nonlinear partial differential equations, International Journal of the Physical Sciences, 6, 29 6706-6716 (2011)
[17] Naher, H., Abdullah, F.A., Akbar, M.A., New travelling wave solutions of the higher dimensional nonlinear partial differential equation by the Exp-function method, Journal of Applied Mathematics, 575387 (2012)
[18] Mohyud-Din, S.T., Noor, M.A., Noor, K.I., Exp-function method for traveling wave solutions of modified Zakharov-Kuznetsov equation, Journal of King Saud University, 22 213-216 (2010)
[19] Ebadi, G., Krishnan, E. V., Labidi, M., Zerrad, E., & Biswas, A., Analytical and numerical solutions to the Davey–Stewartson equation with power-law nonlinearity, Waves in Random and Complex Media, 21(4), 559-590 (2011).
[20] Bruzn, M. S., Gandarias, M. L., Camacho, J. C., Symmetry Analysis and Solutions for a Generalization of a Family of BBM Equations, Journal of Nonlinear Mathematical Physics, 15, 81-90 (2008)
[21] Bruzn, M. S., Camacho, J. C., Gandarias, M. L., Symmetry Analysis and Solutions for a Family of BBM Equations, Nonlinear Evolution Equations and Dynamical Systems, (2007).
[22] Jafari, H., Tajadodi, H., Biswas, A., Homotopy analysis method for solving a couple of evolution equations and comparison with Adomian's decomposition method, Waves in Random and Complex Media, 21 (4), 657-667 (2011)
[23] Bruzn, M. S., Gandarias, M. L., Symmetry reductions and exact solutions of Benjamin-Bona-Mahony-Burgers equation. In 4th Workshop Group Analysis of Differential Equations & Integrable Systems, 45-61 (2009).
[24] Johnpillai, A. G., Kara, A. H., & Biswas, A., Symmetry reduction, exact group-invariant solutions and conservation laws of Benjamin-Bona-Mahoney equation, Applied Mathematics Letters, 26 (3), 376-381 (2013)
[25] Wazwaz, A.M., A new (2+1)-dimensional Korteweg-de-Vries equation and its extension to a new (3+1)-dimensional Kadomtsev-Petviashvili equation, Physica Scripta, 84, 035010.(2011)
[26] Ebadi, G., Kara, A.H., Petkovic, M.D., Biswas, A., Soliton solutions and conservation laws of Gilson-Pickering equation, Waves in Random and Complex Media, 21 (2) 378-385.(2011)
[27] Wang, M., Li, X., Zhang, J., The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Physics Letters A, 372 417-423 (2008)
[28] Bruzn, M. S., Gandarias, M. I., New exact solutions for a Benjamin-Bona-Mahony equation, World Scientific and Engineering Academy and Society (WSEAS), 189-194 (2008).
[29] Zayed, E.M.E., Gepreel, K.A., The (G'/G)-expansion method for finding travelling wave solutions of nonlinear partial differential equations in mathematical physics, Journal of Mathematical Physics, 50 013502-013513 (2009)
[30] Zayed, E.M.E., Al-Joudi, S., Applications of an Extended -Expansion Method to find Exact Solutions of Nonlinear PDEs in Mathematical Physics, Mathematical Problems in Engineering, 768573 (2010)
[31] Ebadi, G., Biswas, A., Application of the (G'/G)-expansion method for nonlinear diffusion equations with nonlinear source, Journal of the Franklin Institute, 347(7), 1391-1398 (2010)

[32] Ebadi, G., Biswas, A., The method and topological soliton solution of the K(m, n) equation, Communications in Nonlinear Science and Numerical Simulation, 16(6), 2377-2382 (2011)
[33] Naher, H., Abdullah, F.A., Akbar, M.A., The (G'/G)-expansion method for abundant traveling wave solutions of Caudrey-Dodd-Gibbon equation, Mathematical Problems in Engineering, 218216. (2011)
[34] Ebadi, G., Biswas, A., The method and 1-soliton solution of the Davey–Stewartson equation. Mathematical and Computer Modelling, 53(5), 694-698 (2011)
[35] Ebadi, G., Krishnan, E. V., Biswas, A., Solitons and cnoidal waves of the Klein–Gordon–Zakharov equation in plasmas, Pramana, 79(2), 185-198 (2012)
[36] Ebadi, G., Yildirim, A., Biswas, A., Chiral solitons with Bohm potential using method and exp-function method, Romanian Reports in Physics, 64(2), 357-366 (2012)
[37] Ebadi, G., Mojaver, A., Johnson, S., Kumar, S., Biswas, A., Dynamics of dispersive topological solitons and its perturbations, Indian Journal of Physics, 86(12), 1115-1129 (2012)
[38] Bekir, A., Aksoy, E., Exact solutions of shallow water wave equations by using the (G'/G)-expansion method Waves in Random and Complex Media, 22 (3), 317-331 (2012)
[39] Zhang, H., Application of the (G'/G)-expansion method for the complex KdV equation, Commun Nonlinear Sci Numer Simulat, 15 1700-1704 (2010)
[40] Liu, X., Tian, L., Wu, Y., Application of (G'/G)-expansion method to two nonlinear evolution equations, Applied Mathematics and Computation, 217 1376-1384 (2010)
[41] Feng, J., Li, W., Wan, Q., Using (G'/G)-expansion method to seek the travelling wave solution of Kolmogorov-Petrovskii-Piskunov equation, Applied Mathematics and Computation, 217, 5860-5865 (2011)
[42] Zhang, J., Jiang, F., Zhao, X., An improved (G'/G)-expansion method for solving nonlinear evolution equations, International Journal of Computer Mathematics, 87, 8 1716-1725.(2010)
[43] Zhao, Y.M., Yang, Y.J., Li, W., Application of the improved (G'/G)-expansion method for the Variant Boussinesq equations, Applied Mathematics Sciences, 5, 58 2855-2861 (2011)
[44] Hamad, Y.S., Sayed, M., Elagan, S.K., El-Zahar, E.R., The improved (G'/G)-expansion method for solving (3+1)-dimensional potential-YTSF equation, Journal of Modern Methods in Numerical Mathematics, 2, 1-2 32-38 (2011)
[45] Naher, H., Abdullah, F.A., Some new traveling wave solutions of the nonlinear reaction diffusion equation by using the improved (G'/G)-expansion method, Mathematical Problems in Engineering, 871724.(2012)
[46] Naher, H., Abdullah, F.A., The improved (G'/G)-expansion method for the (2+1)-dimensional Modified Zakharov-Kuznetsov equation, Journal of Applied Mathematics, 438928 (2012)
[47] Nofel, T.A., Sayed, M., Hamad, Y.S., Elagan, S.K., The improved (G'/G)-expansion method for solving the fifth-order KdV equation, Annals of Fuzzy Mathematics and Informatics, 3, 1 9-17 (2011)
[48] Yusufoglu, E., Bekir, A., The tanh and the sine-cosine methods for exact solutions of the MBBMN and the Vakhnenko equations, Chaos, Solitons and Fractals, 38 1126-1133 (2008)
[49] Yusufoglu, E., New solitonary solutions for the MBBM Equation using Exp-function method, Physics Lett. A, 372 442-446.(2008)
[50] Taghizadeh, N., Mirzazadeh, M., Exact solutions of modified Benjamin-Bona-Mahony equation and Zakharov-Kuznetsov equation by modified extended tanh method, International Journal of Applied Mathematics and Computation, 3, (2) 151-157 (2011)
[51] Abbasbandy, S., Shirzadi, A., The first integral method for modified Benjamin-Bona-Mahony equation, Commun Nonlinear Sci Numer Simulat, 15 1759-1764 (2010)
[52] Aslan, I., Exact and explicit solutions to some nonlinear evolution equations by utilizing the (G'/G)-expansion method, Applied Mathematics and Computation, 215 857-863 (2009)