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Spaces of Generalized difference Lacunary I-convergent sequences

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Abstract: In this research article, we studied some new generalized difference strongly summable lacunary I-convergent n-normed sequence spaces related to ℓ_p spaces which are defiend by Orlicz functions. Some results involved with these spaces are also investigated and studied. We also give some relations related to these sequence spaces.

Keywords: Lacunary sequence, n-norm, Orlicz function; Difference operator; Ideal Convergence; de la Vallè Poussin mean.

1 Introduction

The concept of the crisp set sequence space $m(\phi)$ was initiated by W.L.C. Sargent which was later on studied and investigated from the sequence space point of view by many other mathematicians. Recently, some researchers worked on some matrix classes characterized by taking $m(\phi)$ as one member.

Kostyrko [13] introduced the concept of Ideal convergence as a generalization form of statistical convergence.

A lacunary sequence is defined as an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

For any lacunary sequence $\theta = (k_r)$, the space N_θ is defined as ,

$$N_\theta = \left\{ (x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in J_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

The space N_θ is a BK space with the norm,

$$\|(x_k)\|_\theta = \sup_r h_r^{-1} \sum_{k\in J_r} |x_k|.$$

Note: Throughout this paper, the intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by ϕ_r.

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By an Orlicz function, we mean a function $M : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

A sequence $x \in \ell_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz[9] introduced the following sequence space as,

$$\hat{c} = \{x \in \ell_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + \ldots + x_{m+n}}{m+1}.$$

The notion of difference sequence spaces of crisp sets are defined as $Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\}$, for $Z = \ell_\infty, c$ and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in N$ and later on which was generalized by many other recent mathematicians.

These spaces are Banach spaces, normed by,(see Kizmaz [10])

$$\|x\|_\Delta = |x_1| + \sup_k |\Delta x_k|.$$

2 Definitions and Preliminaries

Let $n \in N$ and X be a real vector space. A real valued function on X^n satisfying the following four properties:

1.$\|(z_1, z_2, \ldots z_n)\|_n = 0$ if and only if $z_1, z_2, \ldots z_n$ are linearly dependent;
2.$\|(z_1, z_2, \ldots z_n)\|_n$ is invariant under permutation;
3.$\|(z_1, z_2, .., .z_{n-1}, \alpha z_n)\|_n = |\alpha| \|z_1, z_2, \ldots z_n\|_n$, for all $\alpha \in R$;
4.$\|(z_1, z_2, \ldots z_{n-1}, x+y)\|_n \leq \|(z_1, z_2, \ldots z_{n-1}, x)\|_n + \|(z_1, z_2, \ldots z_{n-1}, y)\|_n$;

is called an n-norm on X and the pair $(X, \|., .\|_n)$ is called an n-normed space.

The space $m(\phi)$ is defined as,

$$m(\phi) = \left\{ (x_k) \in w : \|x\|_{m(\phi)} = \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty \right\}.$$

The idea of Orlicz function is used to construct the sequence space, (see Lindenstrauss and Tzafriri [11])

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which becomes a Banach space, called as Orlicz sequence space, with the following norm,

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

Let X be a nonempty set. Then a family of sets $I \subseteq 2^X$ (power sets of X) is said to be an ideal if I is additive i.e. $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I, B \subseteq A \Rightarrow B \in I$.

For a lacunary sequence $\theta = (k_r)$, a sequence (x_k) is said to be lacunary I-convergent if for every $\varepsilon > 0$ such that,

$$\left\{ r \in N : h_r^{-1} \sum_{k \in J_r} |x_k - x| \geq \varepsilon \right\} \in I.$$

We write $I_\theta - \lim x_k = x$.

In this article, we define some new generalized difference lacunary I-convergent sequence spaces in n-normed spaces related to ℓ_p-space by using Orlicz function. We will also introduce and examine certain new sequence spacIes using the above tools.

3 Main Results

Let $u = (u_k)$ be a sequence of real numbers such that $u_k > 0$ for all k, and $\sup\limits_k u_k < \infty$. Also, let I be an admissible ideal of N and M be an orlicz function. In this article, we have introduced the following sequence space as,

$$(m(M, \phi, \Delta_p^q, u, \theta)^I, \|, \ldots \|_n)$$
$$= \left\{ x : \forall \varepsilon > 0 \left(r \in N : \tfrac{1}{h_r} \left(\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} \left(M \left(\frac{\|\Delta_p^q t_{km}(x) - L, z_1, z_2, \ldots z_{n-1}\|_n}{\rho} \right) \right)^{u_k} \right) \geq \varepsilon \right) \in I \right\}.$$

for some $\rho > 0, \forall z_1, z_2, \ldots z_{n-1} \in X$.

Particular cases: If we take $u_k = 1$, for all k, we have,

$$(m(M, \phi, \Delta_p^q, \theta)^I, \|, \ldots \|_n)$$
$$= \left\{ x : \forall \varepsilon > 0 \left(r \in N : \tfrac{1}{h_r} \left(\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\|\Delta_p^q t_{km}(x) - L, z_1, z_2, \ldots z_{n-1}\|_n}{\rho} \right) \right) \geq \varepsilon \right) \in I \right\}.$$

for some $\rho > 0, \forall z_1, z_2, \ldots z_{n-1} \in X$.

Now, if we consider $M(x) = x$, then we can easily obtain:

$$(m(\phi, \Delta_p^q, u, \theta)^I, \|, \ldots \|_n)$$
$$= \left\{ x : \forall \varepsilon > 0 \left(r \in N : \tfrac{1}{h_r} \left(\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} (\|\Delta_p^q t_{km}(x) - L, z_1, z_2, \ldots z_{n-1}\|_n)^{u_k} \right) \geq \varepsilon, \text{ uniformly in } m \right) \in I \right\}.$$

If $x \in (m(M, \phi, \Delta_p^q, u, \theta)^I, \|, \ldots \|_n)$ with

$$\left\{ \tfrac{1}{h_r} \left(\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} \left(M \left(\frac{\|\Delta_p^q t_{km}(x) - L, z_1, z_2, \ldots z_{n-1}\|_n}{\rho} \right) \right)^{u_k} \right) \geq \varepsilon \right\} \in I \text{ as } n \to \infty, \text{ uniformly in } m, \text{ then we write}$$

$x_k \to L \in (m(M, \phi, \Delta_p^q, u, \theta)^I, \|, .. \|_n)$.

The following well known inequality will be used later.

If $0 \le u_k \le \sup u_k = H$ and $C = \max(1, 2^{H-1})$, then

$$|a_k + b_k|^{u_k} \le C\{|a_k|^{u_k} + |b_k|^{u_k}\}, \tag{1}$$

for all k and $a_k, b_k \in C$.

Theorem 1. Let $\liminf_{k\to\infty} u_k > 0$. Then, $x_k \to L$ implies $x_k \to L \in (m(M,\phi,\Delta_p^q,u,\theta)^I, \|,..\|_n)$. Let $\lim_{k\to\infty} u_k = u > 0$. If $x_k \to L \in (m(M,\phi,\Delta_p^q,u,\theta)^I, \|,..\|_n)$, then L is unique.

Proof: Let $x_k \to L$.
By the definition of Orlicz function, we have, for all $\varepsilon > 0$,

$$\left\{\frac{1}{h_r}\left(\sup_{s\ge 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma} M\left(\frac{\|\Delta_p^q t_{km}(x) - L, z_1, z_2, ..z_{n-1}\|_n}{\rho}\right)\right) \ge \varepsilon\right\} \in I.$$

Since $\liminf_k u_k > 0$, it follows that,

$$\left\{\frac{1}{h_r}\left(\sup_{s\ge 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left(M\left(\frac{\|\Delta_p^q t_{km}(x) - L, z_1, ..z_{n-1}\|_n}{\rho}\right)\right)^{u_k}\right) \ge \varepsilon\right\} \in I$$

And consequently, $x_k \to L \in (m(M,\phi,\Delta_p^q,u,\theta)^I, \|,....\|_n)$.

Let $\lim_k s_k = s > 0$. Suppose that,

$$x_k \to L_1 \in (m(M,\phi,\Delta_p^q,u,\theta)^I, \|,..\|_n)$$

$$x_k \to L_2 \in (m(M,\phi,\Delta_p^q,u,\theta)^I, \|,..\|_n)$$

and $(\|L_1 - L_2, z_1, z_2, ...z_{n-1}\|_n)^{u_k} = a > 0$.

Now, using the definitions of inequality and Orlicz function, we have,

$$\frac{1}{h_r}\left\{\sup_{s\ge 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left(M\left(\frac{|L_1 - L_2, z_1, ...z_{n-1}\|_n}{\rho}\right)\right)^{u_k}\right\}$$

$$\le \frac{C}{h_r}\left\{\sup_{s\ge 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left(M\left(\frac{|\Delta_p^q t_{km}(x) - L_1, z_1, ...z_{n-1}\|_n}{\rho}\right)\right)^{u_k}\right\}$$

$$+\frac{C}{h_r}\left\{\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left(M\left(\frac{|\Delta_p^q t_{km}(x)-L_2,z_1,...z_{n-1}\|_n}{\rho}\right)\right)^{u_k}\right\}$$

Since,

$$\left\{r\in N:\frac{1}{h_r}\left(\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left(M\left(\frac{\|\Delta_p^q t_{km}(x)-L_1,z_1,..z_{n-1}\|_n}{\rho}\right)\right)^{u_k}\right)\geq\varepsilon\right\}\in I,$$

and

$$\left\{r\in N:\frac{1}{h_r}\left(\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left(M\left(\frac{\|\Delta_p^q t_{km}(x)-L_2,z_1,..z_{n-1}\|_n}{\rho}\right)\right)^{u_k}\right)\geq\varepsilon\right\}\in I,$$

Hence,

$$\left\{r\in N:\frac{1}{h_r}\left(\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left(M\left(\frac{\|L_1-L_2,z_1,..z_{n-1}\|_n}{\rho}\right)\right)^{u_k}\right)\geq\varepsilon\right\}\in I \tag{2}$$

Further,

$$M\left(\frac{\|L_1-L_2,z_1,z_2,..z_{n-1}\|_n}{\rho}\right)^{u_k}\to M\left(\frac{a}{\rho}\right)^{u}$$

as $k\to\infty$, and therefore,

$$\frac{1}{h_r}\left\{\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{\|L_1-L_2,z_1,..z_{n-1}\|_n}{\rho}\right)^{u_k}\right\}=M\left(\frac{a}{\rho}\right)^{u}. \tag{3}$$

From the above equations ((3) and (4)), it follows that $M\left(\frac{a}{\rho}\right)=0$ and by the definition of an Orlicz function, we have $a=0$. Hence $L_1=L_2$ and this completes the proof.

Theorem 2.

1. Let $0<\inf u_k\leq u_k\leq 1$. Then, $m(M,\phi,\Delta_p^q,u,\theta)^I\subset m(M,\phi,\Delta_p^q,\theta)^I$
2. Let $0<u_k\leq \sup u_k<\infty$. Then, $m(M,\phi,\Delta_p^q,\theta)^I\subset m(M,\phi,\Delta_p^q,u,\theta)^I$.

Theorem 3. The inclusion $m(M,\phi,\Delta_p^{q-1},u,\theta)^I\subset m(M,\phi,\Delta_p^q,u,\theta)^I$ is strict.

In general, $m(M,\phi,\Delta_p^i,u,\theta)^I\subset m(M,\phi,\Delta_p^q,u,\theta)^I$ for all $i=1,2,3,...p-1$ and the inclusion is strict.

Theorem 4. $m(M,\phi,\Delta_p^q,u,\theta)^I$ is a complete linear topological space,with paranorm g, where g is defined by,

$$g(x)=\sum_{m=1}^{pq}\|t_{km}(x)-L,z_1,z_2,..z_{n-1}\|_n$$
$$+\inf\left\{\rho^{\frac{u_k}{H}}:\frac{1}{h_r}\left\{\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left(M\left(\frac{\|\Delta_p^q t_{km}(x)-L,z_1,z_2,..z_{n-1}\|_n}{\rho}\right)\right)^{u_k}\right\}\right\}$$

where $H = \max(1, (\sup_k u_k))$.

Proposition 5. $m(M,\phi,\Delta_p^q,u,\theta)^I \subseteq m(M,\psi,\Delta_p^q,u,\theta)^I$ if and only if $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$, for $0 < p < \infty$.

Proof : First, suppose that $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = K < \infty$, then we have, $\phi_s \leq K\psi_s$.

Now, if $(x_k) \in m(M,\phi,\Delta_p^q,u,\theta)^I$, then

$$\frac{1}{h_r}\left\{\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left(M\left(\frac{\|\Delta_p^q t_{km}(x)-L,z_1,z_2,..z_{n-1}\|_n}{\rho}\right)\right)^{u_k} \geq \varepsilon\right\} \in I$$

$$\Rightarrow \frac{1}{h_r}\left\{\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{K\psi_s}\sum_{k\in\sigma}\left(M\left(\frac{\|\Delta_p^q t_{km}(x)-L,z_1,z_2,..z_{n-1}\|_n}{\rho}\right)\right)^{u_k} \geq \varepsilon\right\} \in I$$

i.e $(x_k) \in m(M,\psi,\Delta_p^q,u,\theta)^I$.

Hence, $m(M,\phi,\Delta_p^q,u,\theta)^I \subseteq m(M,\psi,\Delta_p^q,u,\theta)^I$.

Conversely, suppose that $m(M,\phi,\Delta_p^q,u,\theta)^I \subseteq m(M,\psi,\Delta_p^q,u,\theta)^I$. We should prove that $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = \sup_{s\geq 1}(\eta_s) < \infty$. Suppose that $\sup_{s\geq 1}(\eta_s) = \infty$. Then there exists a subsequence (η_{s_i}) of (η_s) such that $\lim_{i\to\infty}(\eta_{s_i}) = \infty$. Then for $(x_k) \in m(M,\phi,\Delta_p^q,u,\theta)^I$, we have,

$$\frac{1}{h_r}\left\{\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\psi_s}\sum_{k\in\sigma}\left(M\left(\frac{\|\Delta_p^q t_{km}(x)-L,z_1,z_2,..z_{n-1}\|_n}{\rho}\right)\right)^{u_k}\right\}$$

$$\geq \frac{1}{h_r}\left\{\sup_{s\geq 1,\sigma\in\wp_s}\left(\frac{\eta_{s_i}}{\phi_{s_i}}\right)\sum_{k\in\sigma}\left(M\left(\frac{\|\Delta_p^q t_{km}(x)-L,z_1,z_2,..z_{n-1}\|_n}{\rho}\right)\right)^{u_k}\right\} = \infty$$

which implies that $(x_k) \notin m(M,\phi,\Delta_p^q,u,\theta)^I$, a contradiction. This completes the proof.

Corollary 6. $m(M,\phi,\Delta_p^q,u,\theta)^I = m(M,\psi,\Delta_p^q,u,\theta)^I$, if and only if $\sup_{s\geq 1}(\eta_s) < \infty$ and $\sup_{s\geq 1}(\eta_s^{-1}) < \infty$, where $\eta_s = \frac{\phi_s}{\psi_s}$, for $0 < p < \infty$.

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