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Conformable variational iteration method

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Abstract: In this study, we introduce the conformable variational iteration method based on new defined fractional derivative called conformable fractional derivative. This new method is applied two fractional order ordinary differential equations. To see how the solutions of this method, linear homogeneous and non-linear non-homogeneous fractional ordinary differential equations are selected. Obtained results are compared the exact solutions and their graphics are plotted to demonstrate efficiency and accuracy of the method.

Keywords: Conformable fractional derivative, Factional derivative, Fractional differential equations , Variational iteration method.

1 Introduction

Fractional differential equations are located in the center of numerous research area in recent years because of the their commonly appearance in different kinds of applied sciences. Therefore, it was dealt to solve fractional ordinary and partial differential equations by many researches. Some authors such as Poldlubny [1], Schneider and Wyss [2],Beyer and Kemplfe [3] and Mainardi [4], discussed fractional order of ordinary and partial differential equations.

The variational iteration method (VIM) which was developed by He [5–10] are applied to different type differential equations such asv autonomous ordinary differential systems, delay differential and strongly nonlinear equations. Wazwaz [11] solved systems of fourth-order Emden-Fowler type equations by the vim. There are too much application of variational iteration method, some of them in [12–18] The advantage of the method is not specific requirements, such as linearization, small parameters and so on.

Recently, there is too much interest in new well-behaved simple fractional derivative introduced by Khalil et al. [19] named conformable derivative depending just on the limit definition of the derivative. Abdeljawad [20] provide fractional versions of the chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions and Laplace transforms. Cenesiz and Kurt [21,22] give the solutions of time and space fractional heat differential equations by conformable fractional derivative and approximate analytical solution of the time conformable fractional Burger's equation via Homotopy Analysis Method. The solution of space-time fractional Fornberg–Whitham equation in series form is given by Iyiola and Olayinka [23] via using the relatively new method called q-homotopy analysis method. By this new defined conformable drivetive, Acan et al. [24] proposed new type reduced differential transform method called conformable fractional reduced differential transform method (CFRDTM) and gave some applications [25].

In this paper, we introduce conformable variational iteration method *(C-VIM)* based on conformable fractional

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derivative. Some necessary basic definitions are given, then this technique is applied fractional ordinary differential equations *(FODEs)* and results are given comparatively.

2 On the conformable fractional derivative

In this section, we present some basic definitions and important properties of conformable fractional calculus [19,20,26,27].

Definition 1. *[19]. Given a function* $f:[0,\infty)\to$. *Then the conformable fractional derivative of* f *order* α *is defined by Fractional Calculus*

$$(T_\alpha f)(x)=\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{1-\alpha})-f(x)}{\varepsilon}$$

for all $x>0$, $\alpha\in(0,1]$.

Theorem 1. *[19]. Let* $\alpha\in(0,1]$ *and* f,g *be* α*-differentiable at a point* $x>0$. *Then*

(i) $T_\alpha(af+bg)=a(T_\alpha f)+b(T_\alpha g)$ *for* $a,b\in$,
(ii) $T_\alpha(x^p)=px^{p-\alpha}$, *for all* $p\in$,
(iii) $T_\alpha(f(x))=0$, *for all constant functions* $f(x)=\lambda$,
(iv) $T_\alpha(fg)=f(T_\alpha g)+g(T_\alpha f)$,
(v) $T_\alpha(f/g)=\frac{g(T_\alpha f)-f(T_\alpha g)}{g^2}$,
(vi) *If* $f(x)$ *is differentiable, then* $T_\alpha(f(x))=x^{1-\alpha}\frac{d}{dx}f(x)$.

Definition 2. *[19]. Given a function* $f:[a,\infty)\to$. *Then the conformable fractional derivative of* f *order* α *is defined by Fractional Calculus*

$$(T_\alpha^a f)(x)=\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon(x-a)^{1-\alpha})-f(x)}{\varepsilon}$$

for all $x>0$, $\alpha\in(0,1]$.

All properties in Theorem 1 is valid also for Definition 2 when $(x-a)$ is placed instead of x. Conformable fractional derivative of certain functions for Definition 2 is given following as.

(i) $T_\alpha^a\left((x-a)^p\right)=p(x-a)^{p-\alpha}\text{forall}p\in R$,
(ii) $T_\alpha^a\left(e^{k\frac{(x-a)^\alpha}{\alpha}}\right)=ke^{k\frac{(x-a)^\alpha}{\alpha}},k\in R$,
(iii) $T_\alpha^a\left(\sin\left(k\frac{(x-a)^\alpha}{\alpha}+c\right)\right)=k\cos\left(k\frac{(x-a)^\alpha}{\alpha}+c\right),k,c\in R$,
(iv) $T_\alpha^a\left(\cos\left(k\frac{(x-a)^\alpha}{\alpha}+c\right)\right)=-k\sin\left(k\frac{(x-a)^\alpha}{\alpha}+c\right),k,c\in R$,
(v) $T_\alpha^a\left(\frac{(x-a)^\alpha}{\alpha}\right)=1$.

3 Conformable variational iteration method (C-VIM)

We will briefly introduce *C-VIM* for linear and non-linear *FODEs* in this section. For the purpose of illustration of the methodology to the proposed methods, we write the non-linear *FODEs* in the standard operator form

$$T_\alpha u(x)+L\left(u(x)\right)+N\left(u(x)\right)=g\left(x\right) \tag{1}$$

where $L$is a linear operator, N is a nonlinear operator, $g(x)$ is an non-homogeneous term and T_α is conformable fractional derivative of order α with $0<\alpha\leq 1$. To solve differential equation (2.1) via *C-VIM* write the differential equation (1) in the form by the theorem 1,

$$x^{1-\alpha}\frac{du(x)}{dx}+L(u(x))+N(u(x))=g(x). \tag{2}$$

As in classical VIM, the correction functional for equation (2.2) can be constructed as

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(\zeta)[\zeta^{1-\alpha}\frac{du_n(\zeta)}{d\zeta}+L(u_n(\zeta))+N(\tilde{u}_n(\zeta))-g(\zeta)]d\zeta, \tag{3}$$

where λ is a general Lagrangian multiplier and it can be optimally determined by the aid of variational theory [5–10]. Here $\tilde{u}_n$ is a restricted variation [5–10] where $\delta\tilde{u}_n=0$. The first required to determine λ which is the Lagrangian multiplier will be optimally identified. By using the determined Lagrangian multiplier and any selected function u_0, u_{n+1},which is the successive approximations of $u(x)$ for $n\geq 0$, will be obtained readily. Hence, we get the solution as

$$u(x)=\lim_{n\to\infty}u_n(x). \tag{4}$$

4 Numerical applications

Example 1. Considering the following fractional differential equation[26].

$$T_\alpha u(x)+u(x)=0, 0<\alpha\leq 1 \tag{5}$$

with the initial condition

$$u(0)=1. \tag{6}$$

Exact solution of this problem is

$$u(x)=e^{-\frac{x^\alpha}{\alpha}}. \tag{7}$$

For solving by *C-VIM* we obtain the recurrence relation

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(\zeta)[\zeta^{1-\alpha}\frac{du_n(\zeta)}{d\zeta}+u_n(\zeta)]d\zeta, \tag{8}$$

and the following conditions:

$$\begin{aligned}\lambda'(\zeta)|_{\zeta=x}&=0,\\ \lambda(\zeta)|_{\zeta=x}&=-1.\end{aligned} \tag{9}$$

Therefore, the Lagrange multiplier can be identified as $\lambda=-1$. As a result we obtain the following iteration formula.

$$u_{n+1}(x)=u_n(x)-\int_0^x [\zeta^{1-\alpha}\frac{du_n(\zeta)}{d\zeta}+u_n(\zeta)]d\zeta. \tag{10}$$

By the initial condition (6) we write

$$u_0(x)=u(0)=1, \tag{11}$$

now, substituting (9) with (10) respectively, we obtain

$$\begin{aligned} &u_1(x)=1-x \\ &u_2(x,t)=1-2x+\tfrac{x^{2-\alpha}}{2-\alpha}+\tfrac{x^2}{2} \\ &u_3(x,t)=1-3x+\tfrac{3x^{2-\alpha}}{(2-\alpha)}-\tfrac{x^{3-2\alpha}}{3-2\alpha}+\tfrac{3x^2}{2}-\tfrac{x^{3-\alpha}}{2-\alpha}-\tfrac{x^3}{6} \\ &\vdots \end{aligned} \tag{12}$$

when we consider $\alpha=1$, the solution by *C-VIM* is obtained as

$$u_n(x,t)=1-x+\frac{1}{2}x^2-\frac{1}{3!}x^3+\cdots \tag{13}$$

From (12), the *C-VIM* solution

$$u(x)=\lim_{n\to\infty}u_n(x)=e^{-x}. \tag{14}$$

This is also exact solution. Now, we analyze the *C-VIM* and exact solutions graphically for some α values.

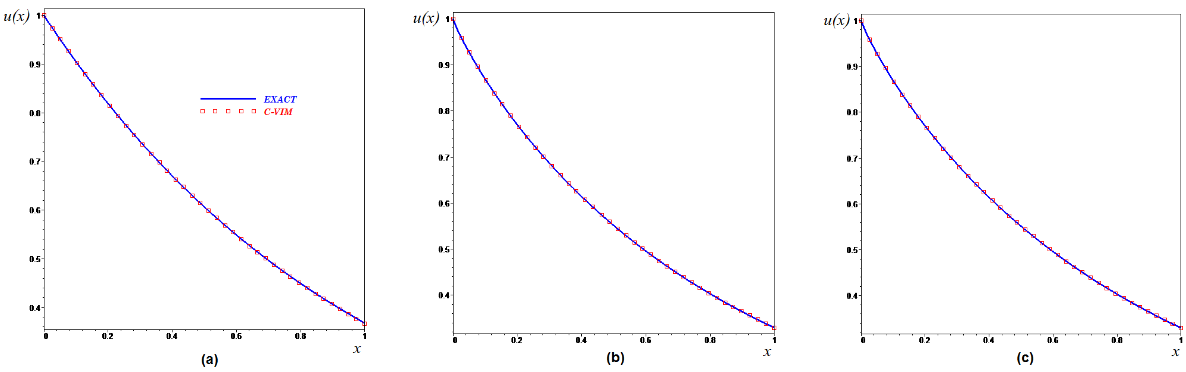

Fig. 1. Comparison of five iteration *C-VIM* solutions with the EXACT solutions for eq. (5): **(a)** $\alpha=1$,**(b)** $\alpha=0.9$,**(a)** $\alpha=0.8$,

Example 2. Considering the following nonlinear fractional Ricatti differential equation[26].

$$T_\alpha u+u^2-1=0, 0<\alpha\le 1 \tag{15}$$

with the initial condition

$$u(0)=0. \tag{16}$$

Exact solution of this problem is

$$u(x)=\frac{e^{\frac{2x^\alpha}{\alpha}}-1}{e^{\frac{2x^\alpha}{\alpha}}+1}. \tag{17}$$

For solving by *C-VIM* we obtain the recurrence relation

$$u_{n+1}(x)=u_n(x)+\int_0^x\lambda(\zeta)[\zeta^{1-\alpha}\frac{du_n(\zeta)}{d\zeta}+u_n^2(\zeta)-1]d\zeta, \tag{18}$$

and the following conditions.

$$\begin{aligned} &\lambda'(\zeta)|_{\zeta=x}=0, \\ &\lambda(\zeta)|_{\zeta=x}=-1. \end{aligned} \tag{19}$$

Therefore, the Lagrange multiplier can be identified as $\lambda = -1$. As a result we obtain the following iteration formula.

$$u_{n+1}(x) = u_n(x) - \int_0^x [\zeta^{1-\alpha}\frac{du_n(\zeta)}{d\zeta} + u_n^2(\zeta) - 1]d\zeta, \tag{20}$$

By the initial condition (15) we write

$$u_0(x) = u(0) = 0, \tag{21}$$

now, substituting (18) with (19) respectively, we obtain

$$\begin{aligned} &u_1(x) = x \\ &u_2(x,t) = 2x - \tfrac{x^{2-\alpha}}{2-\alpha} - \tfrac{x^3}{3} \\ &\vdots \end{aligned} \tag{22}$$

when we consider $\alpha = 1$, the *C-VIM* solution is obtained as

$$\begin{aligned} &u(x,t) = \lim_{n\to\infty} u_n(x) = x - \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 - \tfrac{17}{315}x^7 \cdots \\ &= \tfrac{e^{2x}-1}{e^{2x}+1} \end{aligned} \tag{23}$$

This is also exact solution. Now, we analyze the *C-VIM* and exact solutions graphically for some αvalues.

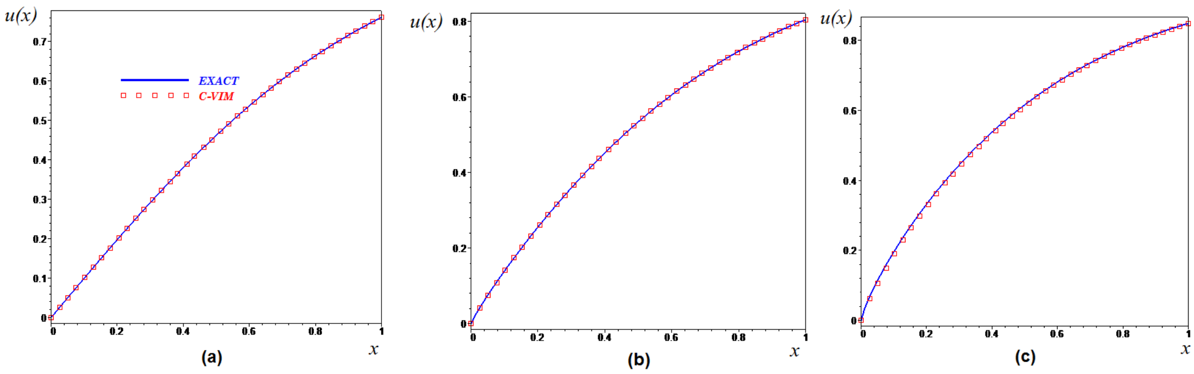

Fig. 2. Comparison of five iteration *C-VIM* solutions with the EXACT solutions for eq. (14): **(a)** $\alpha = 1$,**(b)** $\alpha = 0.9$,**(a)** $\alpha = 0.8$,

5 Conclusion

In this study, the conformable fractional variational iteration method *(C-VIM)* has been successfully applied to two different version of *FODEs*. It is seen in applications, the method have good results with compared exact solutions for linear homogeneous and non-linear non-homogeneous fractional type of ordinary differential equations. The graphics of the solutions are plotted for distinct value of α to see clearly how good *C-VIM* is. As a result, *C-VIM* is useful, effective and has high accuracy to solve fractional ordinary differential equations.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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