Application of Bernoulli wavelet method for numerical solution of fuzzy linear Volterra-Fredholm integral equations

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Abstract: This work, Bernoulli wavelet method is formed to solve nonlinear fuzzy Volterra-Fredholm integral equations. Bernoulli wavelets have been Created by dilation and translation of Bernoulli polynomials. First we introduce properties of Bernoulli wavelets and Bernoulli polynomials, and then we used it to transform the integral equations to the system of algebraic equations. We compared the result of the proposed method with the exact solution to show the convergence and advantages of the new method. The results got by present wavelet method are compared with that of by collocation method based on radial basis functions method. Finally, the numerical examples explain the accuracy of this method.

Keywords: Bernoulli polynomials, Bernoulli wavelets, Volterra-Fredholm fuzzy integral equations, Integration of the cross product, Product matrix, Coefficient matrix.

1 Introduction

The study of fuzzy integral equations begins with the investigations of Kelva [1] and Seikkala [2] for the fuzzy Volterra integral equation that is equivalent to the initial value problem for first order fuzzy differential equations, where the Banach's fixed point theorem and the method of successive approximations are applied in the problem of the existence and uniqueness the solutions. The main problems that arise for fuzzy integral equations are: the existence and uniqueness of the solution, and the construction of numerical methods to approximate it.

Many researchers have focused their interest on this field and published many articles which are available in literature. Many analytical methods like Adomian decomposition method [5], homotopy analysis method [6], and homotopy perturbation method [7] have been used to solve fuzzy integral equations. There are available many numerical techniques to solve fuzzy integral equations. The method of successive approximations [8,9], quadrature rule [10], Nystrom method [11], Lagrange interpolation [12], Bernstein polynomials [13], Chebyshev interpolation [14], Legendre wavelet method [15], sinc function [16], residual minimization method [17], fuzzy transforms method [18], and Galerkin method [19] have been applied to solve fuzzy integral equations numerically. we introduce fuzzy linear Volterra-Fredholm integral equation is introduced.

The rest of the paper has been organized as follows: In section 2, we present some preliminaries and notations useful for fuzzy integral equations. In section 3, we discuss the properties of Bernoulli wavelets and function approximation. In section 4, we establish the method for solving Volterra-Fredholm integral equation. Section 5 deals with the illustrative example which show the efficiency of the presented method.

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2 Preliminaries of fuzzy integral equation

Definition 1. *(See Ref. [20].) A fuzzy number u is represented by an ordered pair of functions* $(\underline{u}(r), \bar{u}(r)); 0 \leq r \leq 1$ *which satisfying the following properties.*

(I) $\underline{u}(r)$ *is a bounded monotonic increasing left continuous function.*
(II) $\bar{u}(r)$ *is a bounded monotonic decreasing left continuous function.*
(III) $\underline{u}(r) \leq \bar{u}(r), 0 \leq r \leq 1$.

For arbitrary $u(r) = (\underline{u}(r), \bar{u}(r)), v(r) = (\underline{v}(r), \bar{v}(r))$, *and* $k > 0$ *we define addition* $(u + v)$ *and scalar multiplication by k as.*

(a) $\underline{(u+v)}(r) = \underline{u}(r) + \underline{v}(r)$
(b) $\overline{(u+v)}(r) = \bar{u}(r) + \bar{v}(r)$
(c) $\underline{ku}(r) = k\underline{u}(r), \overline{ku}(r) = k\bar{u}(r).$

Remark. (See Ref. [21].) If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists. Also

$$\underline{(\int_a^b f(t;r)dt)} = \int_a^b \underline{f}(t;r)dt,$$

$$\overline{(\int_a^b f(t,r)dt)} = \int_a^b \bar{f}(t;r)dt.$$

Definition 2. *([22]) A fuzzy number is a function such as* $u : R \to [0;1]$ *satisfying the following properties.*

(i) *u is normal, i.e.* $\exists x_0 \in R$ *with* $u(x_0) = 1$,
(ii) *u is a convex fuzzy set i.e.* $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\} \forall x, y \in R, \lambda \in [0,1]$,
(iii) *u is upper semi-continuous on R,*
(iv) $\{x \in R : u(x) > 0\}$ *is compact, where A denotes the closure of A.*

The set of all fuzzy real numbers is denoted by E. Obviously $R \subset E$. *Here* $R \subset E$ *is* $R = \{\chi_x : \chi \text{ is usual real number}\}$. *For* $0 < r \leq 1$, *it is* $[u]_r = \{x \in R; u(x) \geq r\}$ *and* $[u]_0 = \{x \in R; u(x) \geq 0\}$. *Then it is well-known that for any* $r \in [0,1]$, $[u]_r$, *is a bounded closed interval. For* $\tilde{u} = \tilde{v} \in E$ *and , and* $\lambda \in R$ *where sum* $\tilde{u} + \tilde{v}$ *and the means the conventional addition of two intervals (subsets) of* $\lambda [u]_r = \{\lambda x : x \in [u]_r\}$ *means the conventional product between a scalar and a subset of R.*

Definition 3. *([22]) Suppose* $\tilde{u}$ *is a fuzzy number and* $r \in [0,1]$. *Then the r-cut representation of* $\tilde{u}$ *is the pair of functions L(r) and R(r) both form [0;1] to R defined respectively, by*

$$L(r) = \inf\{\langle x \mid x \rangle \in [u]_r\}; \text{ if } r \in (0;1] = \inf\{\langle x \mid x \rangle \in \sup p(\tilde{u})\}; \text{ if } r = 0$$

and

$$R(r) = \sup\{\langle x \mid x \rangle \in [u]_r\}; \text{ if } r \in (0;1] = \sup\{\langle x \mid x \rangle \in \sup p(\tilde{u})\}; \text{ if } r = 0.$$

Definition 4. *([22]) A fuzzy number vector* $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_n)^t$ *given by* $\tilde{x}_i = [\underline{\tilde{x}}_i(r), \ldots, \bar{x}_i(r)]$, $1 \leq i \leq n$, $1 \leq r \leq n$ *is called the solution of Volterra-Fredholm integral equation if*

$$\underline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \underline{a_{ij}x_j} = \underline{b_i}, \quad \overline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \overline{a_{ij}x_j} = \overline{b_i}.$$

Definition 5. *Let* $f : R \to E$ *be a fuzzy function (where E is a subset of a Banach space) and* $t_0 \in R$. *The derivative* $f'(t_0) =$ *of f at a point* t_0 *is defined by*

$$f'(t_0) = \lim_{h \to 0^+} \frac{f(t_0 + h) - f(t_0)}{h},$$

provided that this limit taken with respect to the metric D, exists and $h > 0$ be sufficiently small parameter. The elements $f(t_0 + h)$ and $f(t_0)$ in the above equation are in Banach space $B = \bar{C}[0,1] \times \bar{C}[0,1]$.

Thus, if $f(t_0 + h) = (\bar{a}, \underline{a})$ and $f(t_0) = (\underline{b}, \bar{b})$, then $f(t_0 + h) - f(t_0) = (\underline{a} - \underline{b}, \bar{a} - \bar{b})$.

Clearly $[f(t_0 + h) - f(t_0)]/h$ may not be a fuzzy number for all h. However, if it approaches $f'(t_0)$ (in B) and $f'(t_0)$ is also a fuzzy number (in E) this number is the fuzzy derivative of $f(t)$ at t_0. In this case, if $f = (\underline{f}, \overline{f})$ $f'(t_0) = (\underline{f'}(t_0), \overline{f'}(t_0))$ where $(\underline{f'}, \overline{f'})$ are classic derivative of $(\underline{f}, \overline{f})$, respectively and $t_0 \in R$.

3 Wavelets and Bernoulli wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter ***a*** and the translation parameter ***b*** vary continuously, we have the following family of continuous wavelets as $\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi(\frac{t-b}{a})$,

$$a, b \in \mathrm{R}, a \neq 0 \tag{1}$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = n b_0 a_0^{-k}$, $a_0 > 1, b_0 > 0$ and n and k are positive integers, we have the following family of discrete wavelets: $\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^k t - n b_0)$,

$$n, k \in Z^+ \tag{2}$$

where $\psi_{k,n}(t)$ forms a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2, b_0 = 1$, then $\psi_{k,n}(t)$ form an orthonormal basis.

Bernoulli wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments, where $n = 1, 2, ..., 2^{k-1}, k \in Z^+, m$ is the order of Bernoulli polynomials and t is normalized time. They are defined on the interval [0,1) as [14].

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{\beta}_m(2^{k-1}t - n + 1) & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}} \\ 0, & otherewise \end{cases} \tag{3}$$

with

$$\tilde{\beta}_m(t) = \begin{cases} 1, & m = 0 \\ \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}} \beta_m(t), & m > 0 \end{cases} \tag{4}$$

where $m = 0, 1, ..., M-1$ and $n = 1, 2, ..., 2^{k-1}$.

The coefficient $\frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}$ is for the orthonormality, the dilation parameter is $a = 2^{-(k-1)}$ and translation parameter is $b = (n-1)2^{-(k-1)}$.

Here $\beta_m(t)$ are the well-known m^{th} order Bernoulli polynomials which are defined on the interval [0, 1], and can be determined with the aid of the following explicit formula [15].

$$\beta_m(t) = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-i} t^i, \tag{5}$$

where are α_i. $i = 0, 1, ..., m$ are Bernoulli numbers.

The first four such polynomials, respectively, are $\beta_0(t) = 1$, $\beta_1(t) = t - \frac{1}{2}$, $\beta_2(t) = t^2 - t + \frac{1}{6}$, $\beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$.

Bernoulli polynomials satisfy the following formula [24].

$$\int_0^1 \beta_n(t)\beta_m(t)\,dt = (-1)^{n-1}\frac{m!\,n!}{(m+n)!}\beta_{m+n}\, m, n \geq 1. \tag{6}$$

3.1 Properties of Bernoulli's polynomial

Properties of Bernoulli polynomials are given as follows [24].

(1) $\beta_m(1-t) = (-1)^m\beta_m(t), m \in Z^+$
(2) $\beta_m^/(t) = m\beta_{m-1}(t), m \in Z^+$
(3) $\int_0^1 \beta_m(t)\beta_n(t)dt = (-1)^{m-1}\frac{m!\,n!}{(m!+n!)}\alpha_{m+n}(t), m, n \geq 1.$
(4) $\int_0^1 |\beta_m(t)|\,dt < 16\frac{m!}{(2\pi)^{m+1}}\alpha_{m+n}(t), m \geq 0.$
(5) $\int_a^x \beta_m(t)dt = \frac{\beta_{m+1}(x) - \beta_{m+1}(a)}{m+1}.$
(6) $\sup_{t \in [0,1]} |\beta_{2m}(t)| = |\alpha_{2m}|.$
(7) $\sup_{t \in [0,1]} |\beta_{2m+1}(t)| \leq \frac{2m+1}{4}|\alpha_{2m}|.$

3.2 Properties of Bernoulli number

The sequence of Bernoulli numbers $(\alpha_m)_{m \in N}$ satisfying the following properties [24],

(1) $\alpha_{2m+1} = 0, \alpha_{2m} = \beta_{2m}(1).$
(2) $\beta_m(1/2) = (2^{1-m} - 1)\alpha_m.$
(3) $\alpha_m = -\frac{1}{m+1}\sum_{k=0}^{m-1}\binom{m+1}{k}\alpha_k.$

3.3 Function approximation by using Bernoulli wavelet method

Any function $y(t)$ which is square integrable in the interval $[0, 1)$ can be expanded in a Bernoulli wavelet method (BWM) as.

$$\underline{y}(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} y_{n,m}b_{n,m}(t) = B^T(t)Y \tag{7}$$

$$Y = \frac{(y(t), b_{n,m}(t))}{(b_{n,m}(t), b_{n,m}(t))} \tag{8}$$

In (8), (. , .) denotes the inner product. If the infinite series in (7) is truncated, then (7) can be rewritten as

$$Y = [y_{1,0}, y_{1,1}, ..., y_{1,M-1}, y_{2,0}, ..., y_{2,M-1}, ..., y_{2^{k-1},0}, ..., y_{2^{k-1},M-1}]^T$$

$$B(t) = [b_{1,0}(t), b_{1,1}(t), ..., b_{1,M-1}(t), b_{2,0}(t), ..., b_{2,M-1}(t), ..., b_{2^{k-1},0}(t), ..., b_{2^{k-1},M-1}(t)]^T.$$

Therefore we have

$$Y^T < B(t), B(t) >=< u(t), B(t) >$$

then

$$Y = D^{-1} < u(t), B(t) >,$$

where

$$D =< B(t), B(t) >= \int_0^1 B(t).B^T(t)\,dt = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & D_M \end{pmatrix}. \tag{9}$$

Then by using (7) $D_i(i = 1, 2, ..., M)$ is defined as follows.

$$(D_n)_{i,j+1} = \int_{\frac{i-1}{2^{(k-1)}}}^{\frac{i}{2^{(k-1)}}} B_{i,n}(2^{k-1}t - i + 1)B_{j,n}(2^{k-1}t - i + 1)dx = \frac{1}{2^{k-1}} \int_0^1 B_{i,n}(t)B_{j,n}(t)\,dt \tag{10}$$

We can also approximate the function $k(x,t) \in L[0,1]$ as follows:

$$k(x,t) \approx B^T(x)KB(t), \tag{11}$$

where K is an $2^{k-1} \cdot M$ matrix that we can obtain as follows.

$$K = D^{-1} < B(x) < k(x,t), B(t) >> D^{-1} \tag{12}$$

3.4 Integration of Bernoulli wavelet functions

In Bernoulli wavelet functions analysis for a dynamic system, all functions need to be transformed into BWM functions. The integration of BWM functions should be expandable into BWM functions with the coefficient matrix P. These ideas come from papers of Chen et al. [25].

We can approximate function with this base. For example for $k = 2$ and $M = 2$.

$$\psi_{1,0}(t) = \begin{cases} \sqrt{2}, & 0 \le x < \frac{1}{2} \\ 0, & otherewise \end{cases}, \psi_{2,0}(x) = \begin{cases} \sqrt{2}, & \frac{1}{2} \le x < 1 \\ 0, & otherewise \end{cases}$$

$$\psi_{1,1}(x) = \begin{cases} \sqrt{6}(4t - 1), & 0 \le x < \frac{1}{2} \\ 0, & otherewise \end{cases}, \psi_{2,1}(x) = \begin{cases} \sqrt{6}(4t - 3), & \frac{1}{2} \le x < 1 \\ 0, & otherewise \end{cases}$$

$$\psi_{(4\times1)}(t) \equiv [\psi_{10}(t), \psi_{20}(t), \psi_{11}(t), \psi_{21}(t)]^T$$

$$\psi_{(4\times1)}(t) \equiv [\psi_{10}(t), \psi_{20}(t), \psi_{11}(t), \psi_{21}(t)]^T$$

$$\int_0^t B_{(2^{k-1}\cdot M)}(\tau)\,d(\tau) \approx P_{2^{k-1}\cdot M \times 2^{k-1}\cdot M}B_{(2^{k-1}\cdot M)}(t), t \in [0,1), \tag{13}$$

where the $2^{k-1} \cdot M$-square matrix P is called the operational matrix of integration, and $\psi_{(2^{k-1} \cdot M)}(t)$ is defined in Eq. (3). A subscript $2^{k-1} \cdot M \times 2^{k-1} \cdot M$ of P denotes its dimension and P is the operational matrix of integration and can be obtained as.

$$P_{(2^{k-1} \cdot M) \times (2^{k-1} \cdot M)} = \begin{bmatrix} 0.25000 & 0.49999999 & 0.14433756 & 3.53554 \cdot 10^{-30} \\ 0 & 0.24999999 & 0 & 0.144337567 \\ -0.1443375 & -3.5355339 \cdot 10^{-30} & -1.02062 \cdot 10^{-10} & -6.909032 \cdot 10^{-30} \\ 0 & -0.14433756717 & 0 & 1.0206207 \cdot 10^{-10} \end{bmatrix}$$

The integration of the cross product of two BWM function vectors can be obtained as,

$$D = \int_0^1 B_{(2^{k-1} \cdot M)}(t) B^T_{(2^{k-1} \cdot M)}(t) \, d(t) \approx \begin{bmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & L \end{bmatrix} \tag{14}$$

where L is an $2^{k-1} \cdot M$ diagonal matrix given by

$$D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{15}$$

Eqs. (7-15) are very important for solving Volterra- Fredholm-Integral equation of the second kind problems, because the D and P matrix can increase the calculating speed, as well as save the memory storage.

4 Solution of Volterra- Fredholm integral equation via Bernoulli wavelet method

Consider the following Volterra- Fredholm integral equation of the form:

$$y(x) = f(x) + \int_0^1 k_1(x,t) y(t) \, dt + \int_0^x y(t) \, dt \tag{16}$$

where y and f, are fuzzy functions. Let

$$y(x) = [\underline{y}(x,r), \bar{y}(x,r)],$$

$$f(x) = [\underline{f}(x,r), \bar{f}(x,r)],$$

Eq. (9.2), in crisp sense, converted into a system as

$$\underline{y}(x,r) = \underline{f}(x,r) + \int_0^1 k_1(x,t) \underline{y}(t,r) \, dt + \int_0^x \underline{y}(t,r) \, dt$$

$$\bar{y}(x,r) = \bar{f}(x,r) + \int_0^1 k_1(x,t) \bar{y}(t,r) \, dt + \int_0^x \bar{y}(t,r) \, dt \tag{17}$$

where

$$k_1(x,t) \underline{y}(t,r) = \begin{cases} k_1(x,t) \underline{y}(t,r) & k_1(x,t) \geq 0 \\ k_1(x,t) \bar{y}(t,r) & k_1(x,t) < 0 \end{cases}$$

and

$$k_1(x,t)\bar{y}(t,r) = \begin{cases} k_1(x,t)\bar{y}(t,r) & k_1(x,t) \geq 0 \\ k_1(x,t)\underline{y}(t,r) & k_1(x,t) < 0 \end{cases}$$

Approximate $\underline{y}(x,r)$, $\bar{y}(x,r)$, $k_1(x,t)$, $\underline{f}(x,r)$, $\bar{f}(x,r)$ and $k_1(x,t)$ as follows.

$$
\begin{aligned}
k_1(x,t) &\approx B^T(x)K_1 B(t), &\quad \underline{y}(x,r) &\approx B^T(x)Y_1 B(r), \\
\bar{y}(x,r) &\approx B^T(x)Y_2 B(r), &\quad \underline{f}(x,r) &\approx B^T(x)F_1 B(r), \\
\bar{f}(x,r) &\approx B^T(x)F_2 B(r),
\end{aligned}
\tag{18}
$$

with substituting above equations into in Eq. (17)

$$B^T(x)Y_1 B(r) = B^T(x)F_1 B(r) + \int_0^1 B^T(x)K_1 B(t) B^T(t)Y_1 B(r)\,dt + \int_0^x B(t)Y_1^T B(r)dt$$

$$B^T(x)Y_2 B(r) = B^T(x)F_2 B(r) + \int_0^1 B^T(x)K_1 B(t) B^T(t)Y_2 B(r)\,dt + \int_0^x B(t)Y_2^T B(r)dt \tag{19}$$

$$B^T(x)Y_1 B(r) = B^T(x)F_1 B(r) + B^T(x)K_1 \int_0^1 B(t) B^T(t)Y_1 B(r)\,dt + Y_1^T \int_0^x B(t) B(r)dt$$

$$B^T(x)Y_2 B(r) = B^T(x)F_2 B(r) + B^T(x)K_1 \int_0^1 B(t) B^T(t)Y_2 B(r)\,dt + Y_2^T \int_0^x B(t) B(r)dt. \tag{20}$$

Applying Eqs. (10), (12) and (20) to Eq. (20) and Eq.(20) becomes

$$
\begin{aligned}
B^T(x)Y_1 &= B^T(x)F_1 + B^T(x)K_1 DY_1 + Y_1^T PB(x) \\
B^T(x)Y_2 &= B^T(x)F_2 + B^T(x)K_1 DY_2 + Y_2^T PB(x).
\end{aligned}
\tag{21}
$$

In order to find Y we collocate Eq. (21) in $M \cdot 2^k$ nodal points of Newton-Cotes [26] as

$$t_i = \frac{2i-1}{M \cdot 2^k}. \tag{22}$$

From Eqs. (21) we have a system of $M \cdot 2^k$ linear equations and $M \cdot 2^k$ unknowns. After solving above linear system, we can achieve the unknown vectors Y. The required approximated solution $y(x)$ for Volterra–Fredholm integral Eq. (16) can be obtained by using Eqs.(21) as follows.

$$\underline{y}(x,r) = \underline{f}(x,r) + B^T(x)K_1 DY_1 + Y_1^T PB(x) \tag{23}$$

$$\bar{y}(x,r) = \bar{f}(x,r) + B^T(x)K_1 DY_2 + Y_2^T PB(x) \tag{24}$$

5 Illustrative numerical example

We applied the presented schemes to the following Volterra-Fredholm Integral equation of second kind. For this purpose, we consider the following example.

Consider the following linear Volterra- Fredholm Integral equation

$$y(x) = f(x) + \int_0^1 (x+t)\,y(t)\,dt + \int_0^x y(t)\,dt$$

$$f(x,r) = [\underline{f}(t,r), \bar{f}(t,r)] = x^2 - \frac{1}{3}x^3 - \frac{1}{4} - \frac{x}{3} + [0.9 + 0.1\,r, 1.25 - 0.25\,r] - [3.6 + 0.4\,r, 5 - r]x. \tag{25}$$

If we solve (25) for $y(x)$ directly, the analytic solution can be shown to be

$$y(x,r) = [\underline{y}(x,r), \bar{y}(x,r)] = x^2 + [1.8 + 0.2\,r, 2.5 - 0.5\,r], \quad r \in [0,1],\ x \in [0,1].$$

The above problem has been solved by Bernoulli wavelet method. The comparison between the BWM solution and the analytic solution for $x \in [0,1)$ and $t \in [0,1)$ is shown in Fig. 1,2 for $M = 4$ and $k = 3$, then the results are compared with that of obtained by collocation method based on radial basis functions method. We take $x = 0, 0.2, 0.4, 0.6, 0.8, 1$ and $r = 0$ and calculate the absolute errors as $|e_r| = |y(x,r) - y^*(x,r)|$. This comparison is presented in the Table 1. The approximation solutions of $\underline{y}(x,r)$ and $\bar{y}(x,r)$ for $r = 0, 0.1, 0.3, 0.7, 0.9$ are shown in Fig.1 and Fig.2. Better approximation is expected by increasing the order of the Bernoulli polynomials.

Table 1: Comparison of numerical solutions for $\underline{y}(x,r), \bar{y}(x,r)$ in Example 1 at r =0 in [23].

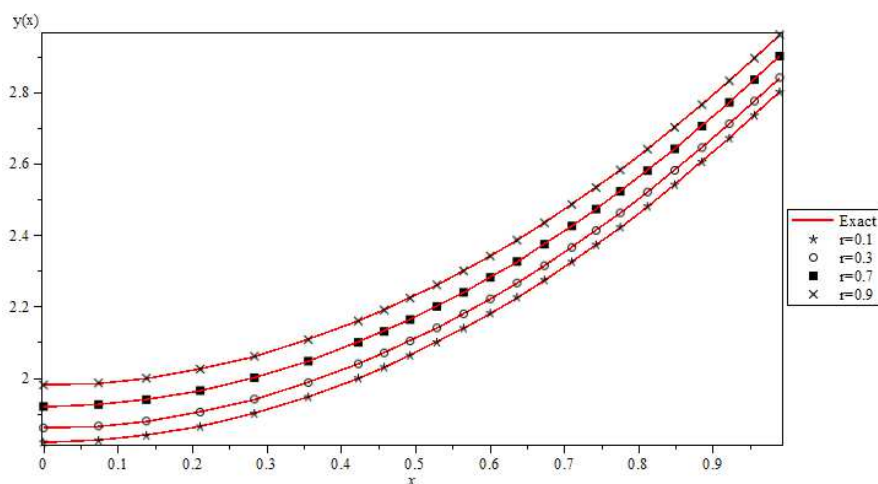| x	Absolute error using BWM (proposed method)		Absolute error using collocation method based on radial basis functions	
0	$\left|y^e - \underline{y}^a\right|$	$\left|\bar{y}^e - \bar{y}^a\right|$	$\left|y^e - \underline{y}^a\right|$	$\left|\bar{y}^e - \bar{y}^a\right|$
0	0	0	0.10	0.82
0.2	0	0	0.44	0.86
0.4	0	$9.6 * 10^{-9}$	0.63	0.99
0.6	$1 * 10^{-9}$	$7.4 * 10^{-9}$	0.72	0.94
0.8	$3.2 * 10^{-8}$	$5.3 * 10^{-8}$	0.40	0.79
1	$4.5 * 10^{-6}$	$5.1 * 10^{-6}$	0.70	0.85

Fig. 1: Approximate solution of $\underline{y}(x,r)$ for $r = 0.1, 0.3, 0.7, 0.9$ of Example 1.
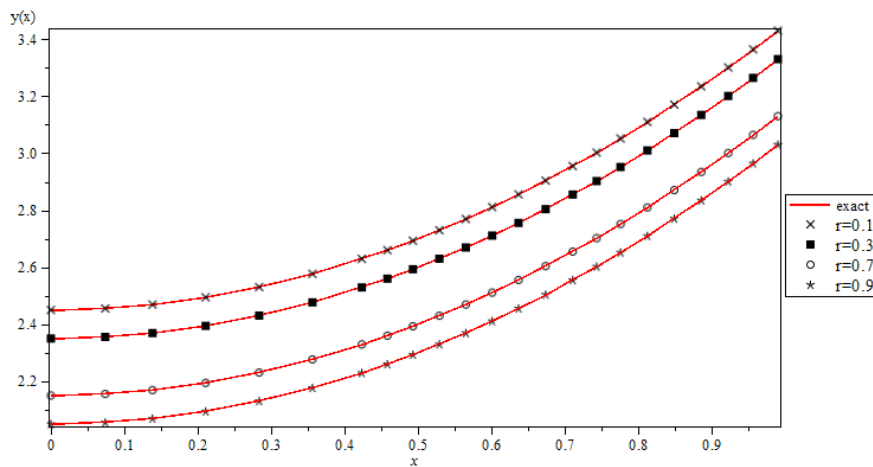
Fig. 2: Approximate solution of $\bar{y}(x,r)$ for $r = 0.1, 0.3, 0.7, 0.9$ of Example 1.

6 Conclusion

In this paper, we proposed an approximation technique to solve fuzzy linear Volterra-Fredholm integral equations. The method is based upon reducing the system into a set of algebraic equations. The generation of this system needs just sampling of functions multiplication and addition of matrices and needs no integration. The matrix D and P are sparse; hence are much faster than other functions and reduces the CPU time and the computer memory, at the same time keeping the accuracy of the solution. The numerical example supports this claim. The numerical results obtained by present method is compared with the results obtained by a combination of collocation method and radial basis functions(RBFs) method. From the above table, it manifests that the present Bernoulli wavelet method gives more accurate results than a combination of collocation method and radial basis functions (RBFs) results. Additionally, the computational time of present method is much smaller than that of obtained by a combination of collocation method and radial basis functions (RBFs). Moreover, the absolute error improves by increasing the order of the Bernoulli polynomials. Illustrative example is included to demonstrate the validity and applicability of the proposed technique. This example also exhibits the accuracy and efficiency of the present method.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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