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Hermite polynomial approach to determine spherical curves in Euclidean 3-space

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Abstract: In this study we consider a second order linear differential equation characterizing spherical curves according to Frenet frame in Euclidean 3-Space E^3. This equation whose coefficients are related to special function, curvature and torsion, is satisfied by the position vector of any regular arbitrary speed space curve lying on a sphere centered at origin. These type equations are generally impossible to solve analytically and so, for approximate solution we present a numerical method based on Hermite polynomials by using initial conditions. The technique we have used is regard to the matrix forms of Hermite and Taylor polynomials, and their derivatives together with collocation points. Our method reduces the solution of problem to the solution of a system of algebraic equations and the approximate solution is obtained in terms of Hermite and Taylor polynomials. Also, a geometrical example is performed to illustrative the validity and applicability of the method.

Keywords: Curves in Euclidean Space, spherical curves, Hermite polynomials and series, matrix and collocation methods.

1 Introduction

The concept of curve defined by Euler in the plane was moved by Fujiwara to the three-dimensional Euclidean space [8, 9]. Shortly after this work, the space curves of constant breadth was definited on the sphere [4]. Wong gave a global formulation of the condition that a general curve is lie on a sphere [24, 25]. And this formula has taken place in books written on differential geometry as a necessary and sufficient condition for a curve to lie on a sphere. Reuleaux, in his work in the same years, showed kinematic and engineering applications of these curves [20]. The work by Gluck brought into the world of geometry the high-grade curvatures of the curves in Euclidean space [10]. The explicit solvability of the differential equation characterizing a spherical curve is shown and this solution is expressed in terms of the curvature radius and torsion of the curve [5]. Wong reached a clear characterization of spherical curves [25]. Dannon worked on the integral characterization of curves [7]. Sezer gave integral characterizations of a system of differential equations like Frenet obtained for curves of constant breadth and spherical curves and he used these characterizations to determine a criterion for the closeness (periodicity) of a space curve [21, 22].

On the other hand, Akgönüllü obtained solution of linear differential, integral and integro differential equations with Hermite polynomials [2]. In the following, a Hermite collocation method was given for the approximate solution of the high-order linear fredholm integro-differential equations [3]. This Hermite collocation method was used to find approximate solutions of the pantograph equation by Yalçınbaş et al [26]. In addition, this method was used for solving differential-difference equations [11].

In this work, firstly, differential equations characterizing arbitrary-speed regular space curves lying on a sphere centered

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at origin in 3-dimensional Euclidean space were obtained. Then the solutions of these equations were studied. Hermite-collocation method based on Hermite polynomials was used for the solution. Since spherical curves are used to operate various mechanisms, the results of this work can be used in field studies such as mechanical engineering, com design and kinematics. In addition, the solutions obtained for these curves in this study will fill an important gap in the literature.

2 Preliminaries

In this section, we give some basic concepts on differential geometry of space curves and spherical curves in Euclidean 3-space. A differentiable α function, defined as α: $\mathrm{I}\subseteq\mathrm{R}\rightarrow\mathrm{E}^{\mathrm{n}}$ for $\mathrm{I}=\mathrm{t}:\mathrm{a}<\mathrm{t}<\mathrm{b}$, is called a curve defined by coordinate neighborhood (I, α) in E^{n}. The variable $\mathrm{t}\in\mathrm{I}$ is called the parameter of the α curve. If the derivative $\mathrm{d}\alpha(\mathrm{t})/\mathrm{dt}$ of this curve differs from zero everywhere, this curve is called a regular curve [10].

The velocity vector of a regular curve $\alpha(\mathrm{t})$ at $\mathrm{t}=t_0$ is the derivative $\mathrm{d}\alpha(\mathrm{t})/\mathrm{dt}$ evaluated at $\mathrm{t}=t_0$. The velocity vector field is the vector valued function $\mathrm{d}\alpha(\mathrm{t})/\mathrm{dt}$. The speed of $\alpha(\mathrm{t})$ at $\mathrm{t}=t_0$ is the lenght of the velocity vector at $\mathrm{t}=t_0$, $\left|\alpha'(t_0)\right|$ [18].

Theorem 1.*If α is a regular curve in E^3 with $\varkappa>0$, then frenet formulaes*

$$T'=v\varkappa N$$

$$N'=-v\varkappa T+v\tau B$$

$$B'=-v\tau N$$

where $\varkappa$ is the curvature of the curve α, τ is the torsion of the curve α and $v=\alpha'$ is the speed function of the curve α, respectively [19]

3 Hermite matrix-collocation method

In this section, we establish the matrix-collocation method to solve the second-order linear differential equation with variable coefficients, which is in the form

$$\sum_{k=0}^{2} Q_k(t)\,\rho^{(k)}(t)=g(t),\quad 0\leq a\leq t\leq b \tag{1}$$

under the initial conditions

$$\rho^{(k)}(a)=c_k,\quad k=0,1 \tag{2}$$

where $\rho^{(0)}(t)=\rho(t)$ is unknown function; $\mathrm{Q_k}(\mathrm{t})$ and $\mathrm{g}(\mathrm{t})$ are functions defined on the interval [a,b], and c_k, a and b are appropriate constants.

Our aim is to obtain an approximate solution expressed in the series form

$$\rho(t)\cong\rho_N(t)=\sum_{n=0}^{N} a_n\,H_n(t),\quad -\infty<a\leq t\leq b<\infty \tag{3}$$

where a_n , n=0,1,2,...,N are unknown Hermite coefficients; N is chosen any positive integer such that $N \geq 3$; $H_n(t)$, $n = 0,1,2,\ldots,N$, are Hermite polinomials. Hermite polynomials H_n (t) satisfy the Hermite differential equation and are given by the form [2]

$$H_n(t) = \sum_{k=0}^{n/2} (-1)^k \frac{n!}{(n-2k)!k!} 2^{n-2k} t^{n-2k}, \quad -\infty < t < +\infty. \tag{4}$$

Let us assume that the function $\rho(t)$ and its derivatives have the truncated Hermite series expansion of the form (3). Then the solution expressed by (3) and its k th derivatives can be converted to the matrix forms

$$\rho(t) \cong \rho_N(t) = H(t)A \ \ and \ \ \rho^{(k)}(t) \cong \rho_N^{(k)}(t) = H^{(k)}(t)A \tag{5}$$

so that

$$\begin{aligned}
H(t) &= \left[\ H_0(t)\ H_1(t)\ H_2(t)\ \ldots\ H_N(t)\ \right] \\
H^{(k)}(t) &= \left[\ H_0^{(k)}(t)\ H_1^{(k)}(t)\ H_2^{(k)}(t)\ \ldots\ H_N^{(k)}(t)\ \right] \\
A &= \left[a_0\ a_1\ a_2\ \ldots\ a_N\right]^T.
\end{aligned}$$

Now we clearly write the matrix form **H**(t), by means of Hermite polynomials defined by (4), as

$$H(t) = T(t)F \tag{6}$$

where $T(t) = [1\ t\ t^2\ \ldots\ t^N]$ and for even values of N

$$\text{F} = \begin{vmatrix}
\frac{(-1)^0 0!2^0}{0!0} & 0 & \frac{(-1)^1 2!2^0}{1!0!} & \cdots & \frac{(-1)^1 2!2^0}{1!0!} \\
0 & \frac{(-1)^0 1!2^1}{0!1!} & 0 & \cdots & 0 \\
0 & 0 & \frac{(-1)^0 2!2^2}{0!2!} & \cdots & \frac{(-1)^{\left(\frac{N}{2}-1\right)} 0!2^0}{\left(\frac{N}{2}-1\right)!2!} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{(-1)^0 N!2^N}{0!N!}
\end{vmatrix}$$

for odd values of N

$$\text{F} = \begin{vmatrix}
\frac{(-1)^0 0!2^0}{0!1!} & 0 & \frac{(-1)^1 2!2^0}{1!0!} & \cdots & 0 \\
0 & \frac{(-1)^0 1!2^1}{0!1!} & 0 & \cdots & \frac{(-1)^{\left(\frac{N-1}{2}\right)} 0!2^1}{\left(\frac{N-1}{2}\right)!1!} \\
0 & 0 & \frac{(-1)^0 2!2^2}{0!2!} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{(-1)^0 N!2^N}{0!N!}
\end{vmatrix}$$

By using 5 and 6, we have the matrix relations

$$\rho_N(t) = T(t)FA \ \ and \ \ \rho_N^{(k)}(t) = T^{(k)}(t)FA. \tag{7}$$

Besides it is clearly seen that the recurrence relation between the matrix $T(t)$ and $T^{(k)}(t)$ is

$$T^{(k)}(t) = T(t)B^k, \quad k = 0,1,2 \tag{8}$$

where

$$B^0 = \begin{bmatrix} 1\;0\;0 & \ldots & 0 \\ 0\;1\;0 & \ldots & 0 \\ 0\;0\;1 & \ldots & 0 \\ \vdots\;\vdots\;\vdots & \ddots & \vdots \\ 0\;0\;0 & \ldots & 1 \end{bmatrix}_{(N+1)x(N+1)} ,B = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(N+1)x(N+1)}$$

From (7) and (8), we obtain the matrix relation, for *k=0,1,2*.

$$\rho_N^{(k)}(\mathrm{t}) = T(t)B^kFA \tag{9}$$

By substituting (9) into Eq.(1), we obtain the matrix equation

$$\sum_{k=0}^{2} Q_k(t)T(t)B^kFA = g(t). \tag{10}$$

On the otherhand, by using the collocations defined by

$$t_i = a + \frac{b-a}{N}i, \quad i = 0,1,2,\ldots,N$$

into Eq.(10), the system of matrix equations as follows:

$$\sum_{k=0}^{2} Q_k(t_i)T(t_i)B^kFA = g(t_i)$$

or the compact form

$$\{\sum_{k=0}^{2} Q_kTB^kF\}A = G \tag{11}$$

where

$$Q_k = \begin{bmatrix} Q_k(t_0) & 0 & \ldots & 0 \\ 0 & Q_k(t_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & Q_k(t_N) \end{bmatrix},$$

$$T = \begin{bmatrix} T(t_0) \\ T(t_1) \\ \vdots \\ T(t_N) \end{bmatrix} = \begin{bmatrix} 1 & t_0 & \ldots & {t_0}^N \\ 1 & t_1 & \ldots & {t_1}^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_N & \ldots & {t_N}^N \end{bmatrix},$$

$$G = \begin{bmatrix} g(t_0) & g(t_1) & \ldots & g(t_N) \end{bmatrix}^T.$$

However the fundamental matrix equation (11) can be expressed in the form

$$WA = G \ \ or \ \ [W;G] \tag{12}$$

where

$$W = [w_{mn}] = \sum_{k=0}^{2} Q_k T B^k F \ ; \quad m,n = 0,1,\ldots,N$$

Note that Eq.(12) indicates a system of (N+1) linear algebraic equations with unknown Hermite coefficients $a_n\ (n = 0,1,\ldots,\ N)$.

On the other hand, by means of the relation (9), the matrix equations for the initial conditions (2) can be written in the forms

$$V_k A = c_k \ \ or \ \ [V_k;c_k], \ \ k = 0,1 \tag{13}$$

where

$$V_k = T(a)B^k F = \left[v_{k_0} \ v_{k_1} \ \ldots \ v_{k_N} \right].$$

Consequently, in order to find the solution of the problem (1)-(2), by replacing the 2 row matixes (13) by the any 2 rows of the augmented matrix (12), we have the new matrix

$$\left[\tilde{W};\ \tilde{G}\right] = A \ or \ (\tilde{W})^{-1} A = \tilde{G} \tag{14}$$

which is a linear algebraic system. In Eq.(14), if $rank\, \tilde{W} = rank\left[\tilde{W};\ \tilde{G}\right] = N+1$, then the coefficients matrix A is uniquelly determined and the solution of the problem (1)-(2) is obtained as

$$\rho_N(t) = H(t)A \text{ or } \rho_N(t) = T(t)FA.$$

4 Obtaining differential equation characterizing spherical curves

In the simplest sense, the concept of spherical curve is defined as ”the curve lying on a sphere” [16]. In this section, we give a necessary and sufficient condition for an arbitrary- speed regular space curve to lie on a sphere centered at origin. Then we obtain that position vector of any arbitrary speed regular space curve lying on a sphere satisfies a second-order linear differential equation with variable coefficients.

Theorem 2. *Let α be a frenet frame curve of class C^4 in E^3 with $\tau \neq 0$ everywhere. Then α lies on a sphere if and only if the following equation holds.*

$$\left[\tau^{-1}\left(\varkappa^{-1}\right)'\right]' + \tau\varkappa^{-1} = 0.$$

Using this differential equation, Dannon observed an equation system Frenet-like for the spherical curves of E^3. Then he proved the accuracy of observation by moving these curves to E^4 [7]. With this in mind, the differential equation characterizing spherical curves in E^3 is obtained as follows.

Let us assume that an arbitrary speed regular α space curve lies on a sphere with radius a centered at C in E^3. In this case, it is obvious that

$$a^2 = \langle C - \alpha(t), C - \alpha(t) \rangle.$$

We can arrange this expression as follows.

$$f(t) = \langle C - \alpha(t), C - \alpha(t) \rangle - a^2 = 0.$$

Now let's get the repetitive differentiations of this f function. We will also use the equations given in theorem 1 for the derivatives of Frenet vector fields. The following equations are obtained:

$$\left\langle C - \alpha(t), (C - \alpha(t))' \right\rangle + \left\langle (C - \alpha(t))', C - \alpha(t) \right\rangle = 0 \ \Rightarrow \ \left\langle (C - \alpha(t))', \alpha(s) \right\rangle = 0$$

$$\langle T(t), C - \alpha(t) \rangle = 0 \tag{15}$$

$$\left\langle T'(t), C - \alpha(t) \right\rangle + \left\langle T(t), (C - \alpha(t))' \right\rangle = 0 \ \Rightarrow \ \langle v\varkappa N(t), C - \alpha(t) \rangle = -v$$

$$\langle N(t), C - \alpha(t) \rangle = -1/\varkappa = \rho \tag{16}$$

$$\left\langle N'(t), C - \alpha(t) \right\rangle + \left\langle N(t), (C - \alpha(t))' \right\rangle = -(1/\varkappa)' = -\rho'$$

$$\langle B(t), C - \alpha(t) \rangle = -\rho'/v\tau \tag{17}$$

Here $C = \alpha(t) + \rho N(t) + (-\rho'/v\tau)B(t)$ is obtained. If C is modified by taking the difference according to t, then

$$(-1/v\tau)\rho'' + (1/v\tau)'\rho' + (v\tau)\rho = 0. \tag{18}$$

This is a second-order linear differential equation with variable coefficient, which characterizes arbitrary-speed spherical curves in Euclidean space-E^3.

5 Solution of differential equation characterizing spherical curves with hermite collocation method

We can arrange the equation (18) characterizing spherical curves as fallows;

$$Q_2(t)\rho'' + Q_1(t)\rho' + Q_0(t)\rho = 0 \tag{19}$$

or

$$\sum_{k=0}^{2} Q_k(t)\rho^{(k)}(t) = g(t) \tag{20}$$

where

$$Q_2(t) = (1/v\tau), Q_1(t) = (1/v\tau)', Q_0(t) = v\tau, g(t) = 0.$$

It is clear that the equations (19) and (20) are equal for g (t) = 0. Let us assume that this equation (20) is an approximate solution on the interval $0 \leq t \leq 2\pi$, under the following initial conditions,

$$\rho^{(k)}(0) = c_k, \ \ k = 0, 1 \tag{21}$$

and in serial form

$$rho\,(t) \cong \rho_N\,(t) = \sum_{n=0}^{N} a_n\, H_n\,(t)\,, \tag{22}$$

and let's take $N = 3$ for the simplicity of our mathematical operations. Where $\rho^{(0)}\,(t) = \rho\,(t)$ is unknown function; $Q_k\,(t)$ and $g\,(t)$ are functions defined on the interval $[0, 2\pi]$, and c_k is appropriate constant, a_n , n=0,1,2,3 are unknown Hermite coefficients, $H_n\,(\mathrm{t})$, $n = 0,1,2,3$ are Hermite polinomials, which are orthogonal on $(-\infty, +\infty)$ and Hermite polynomials are defined as follows

$$H_n\,(t) = \sum_{k=0}^{n/2} (-1)^k \frac{n!}{(n-2k)!k!} 2^{n-2k} t^{n-2k}. \tag{23}$$

Our aim is to obtain an approximate solution expressed in the series form (22). Let us present this approximate solution and its derivatives in matrix form as follows.

$$\rho\,(t) \cong \rho_N(t) = H(t)A \;\; and \;\; \rho^{(k)}\,(\mathrm{t}) \cong \rho_N^{(k)}\,(\mathrm{t}) = H^{(k)}\,(\mathrm{t})A. \tag{24}$$

H (t) and A matrices for $N = 3$ are defined as follows.

$$H\,(t) = \left[\; H_0(t)\; H_1(t)\; H_2(t)\, H_3(t) \;\right]$$
$$H^{(k)}\,(t) = \left[\; H_0^{(k)}(t)\; H_1^{(k)}(t)\; H_2^{(k)}(t)\, H_3^{(k)}(t) \;\right]$$
$$A = [a_0\; a_1\; a_2 \quad a_3]^T.$$

On the other hand, for the $\rho\,(t)$ and its derivatives defined as follows,

$$\rho\,(t) = H(t)B^0 A$$

$$\rho'\,(t) = H(t)B^1 A$$

$$\rho''\,(t) = H(t)B^2 A$$

B^0, B^1 and B^2 matrices are clearly can be written as

$$B^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\; B^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},\; B^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we clearly write the matrix form H(t), by means of Hermite polynomials defined by (23), as

$$H\,(t) = T\,(t)\,F \tag{25}$$

where

$$T\,(t) = \left[\; 1\; t\; t^2 \quad t^3 \;\right].$$

The matrix F for N = 3 is calculated as

$$F = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

By using (24) and (25), we have the matrix relations

$$\rho_N(\mathrm{t}) = \mathrm{T(t)FA} \;\; and \;\; \rho_N^{(k)}(\mathrm{t}) = T^{(k)}(t)FA. \tag{26}$$

Besides, it is known that

$$T^{(k)}(t) = T(t)B^k, \;\; k = 0,1,2$$

in the matrix equation (26) and ρ is found as follows

$$\rho_N^{(k)}(\mathrm{t}) = T(t)B^kFA.$$

Finally, if all these expressions are used in Eq. (20), then the following equation is obtained:

$$Q_2(t)T(t)B^2F + Q_1(t)T(t)B^1F + Q_0(t)T(t)B^0F\,A = G \tag{27}$$

where

$$G = [g_0 \;\; g_1 \;\; g_2 \;\; g_3]^T = [0 \;\; 0 \;\; 0 \;\; 0]^T.$$

On the otherhand, the following matrices are obtained, by using the collocation points defined by $t_0 = 0, \;\; t_1 = 2\pi/3, \;\; t_2 = 4\pi/3, \;\; t_3 = 2\pi$, in Eq.(27).

$$Q_0(t) = \begin{bmatrix} Q_0(0) & 0 & 0 & 0 \\ 0 & Q_0(2\pi/3) & 0 & 0 \\ 0 & 0 & Q_0(4\pi/3) & 0 \\ 0 & 0 & 0 & Q_0(2\pi) \end{bmatrix},$$

$$Q_1(t) = \begin{bmatrix} Q_1(0) & 0 & 0 & 0 \\ 0 & Q_1(2\pi/3) & 0 & 0 \\ 0 & 0 & Q_1(4\pi/3) & 0 \\ 0 & 0 & 0 & Q_1(2\pi) \end{bmatrix},$$

$$Q_2(t) = \begin{bmatrix} Q_2(0) & 0 & 0 & 0 \\ 0 & Q_2(2\pi/3) & 0 & 0 \\ 0 & 0 & Q_2(4\pi/3) & 0 \\ 0 & 0 & 0 & Q_2(2\pi) \end{bmatrix}$$

These matrices briefly can be written for k = 0,1,2 as,

$$Q_k(t) = \begin{bmatrix} Q_k(0) & 0 & 0 & 0 \\ 0 & Q_k(2\pi/3) & 0 & 0 \\ 0 & 0 & Q_k(4\pi/3) & 0 \\ 0 & 0 & 0 & Q_k(2\pi) \end{bmatrix},$$

$$T(t)=\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & (2\pi/3) & (2\pi/3)^2 & (2\pi/3)^3\\ 1 & (4\pi/3) & (4\pi/3)^2 & (4\pi/3)^3\\ 1 & (2\pi) & (2\pi)^2 & (2\pi)^3\end{bmatrix}$$

If we get as

$$Q_2(t)\,T(t)\,B^2F+Q_1(t)\,T(t)\,B^1F+Q_0(t)\,T(t)\,B^0F=W$$

The equation (27) is turned to

$$WA=G\rightarrow\left[\tilde{W};\ \tilde{G}\right]. \tag{28}$$

We calculate the matrix W. Then the equation (28) is written in the form of the increased matrix. Furthermore, to obtain an approximate solution under the initial conditions given by

$$\left.\begin{array}{rl}\rho(0) &= c_0\\ \rho'(0) &= c_1\end{array}\right\} \tag{29}$$

we obtain the matrix equation of conditions as follows

$$\begin{array}{l}\rho(0)\ =T(0)\,B^0FA=\ c_0\\ \rho'(0)=T(0)\,B^1FA=\ c_1\end{array} \tag{30}$$

Where c_0 and c_1 are calculated as fallows.

$$c_0=a_0-2a_2, c_1=2a_1-2a_3$$

Here also, V_0 and V_1 are calculated as follows using

$$V_k=T(0)\,B^kF=\begin{bmatrix}v_{k0}\ v_{k1}\ \ v_{k2}\ v_{k3}\end{bmatrix},\ \ k=0,1$$

equation.

$$V_0=\begin{bmatrix}1\ 0\ -2\ 0\ 12\end{bmatrix},$$

$$V_1=\begin{bmatrix}0\ 2\ 0\ -12\ 0\end{bmatrix}$$

So, the expression in the form of increased matrix of matrix equation of conditions is fallows as;

$$V=\begin{bmatrix}V_0\\ V_1\end{bmatrix}=\begin{bmatrix}1\ \ 0\ -2\ \ 0\ ;c_0\\ 0\ 2\ \ \ 0\ -2\ ;c_1\end{bmatrix}$$

For $P=\begin{bmatrix}c_0\\ c_1\end{bmatrix}$ the following equation is obvious

$$VA=P_{\rightarrow}\left[\tilde{V};\ \tilde{P}\right]. \tag{31}$$

We obtain $W^*A=G^*$ from the equations (28) and (31).

$$\left[W^*;\ G^*\right]=\begin{bmatrix}w_{00} & w_{01} & w_{02} & w_{03} & ;g_0\\ 1 & 0 & -2 & 0 & ;c_0\\ 0 & 2 & 0 & -2 & ;c_1\\ w_{30} & w_{31} & w_{32} & w_{33} & ;g_3\end{bmatrix}$$

Where w_{ij} $(i=0,3,\;\; j=0,1,2,3)$ obtained as fallows;

$$w_{00}=Q_0(0),\;\; w_{01}=2Q_1(0),\quad w_{02}=-2Q_0(\pi)+8Q_2(0),\quad w_{03}=-2Q_1(0),$$

$$w_{30}=Q_0(2\pi),\quad w_{31}=2Q_1(2\pi)+4\pi Q_0(2\pi),$$

$$w_{32}=(16\pi^2-2)Q_0(2\pi)+16\pi Q_1(2\pi)+8Q_2(2\pi)$$

$$w_{33}=(64\pi^3-4\pi)Q_0(2\pi)+(96\pi^2-2)Q_1(2\pi)+96\pi Q_2(2\pi)$$

Furthermore, it is obvious that

$$g_0=w_{00}a_0+w_{01}a_1+w_{02}a_2+w_{03}a_3=0$$

$$g_3\;=w_{30}a_0+w_{31}a_1+w_{32}a_2+w_{33}a_3=0$$

from equation (27). Thus the elements of unknowns matrix $A=W^{*^{-1}}G^*$ is obtained.

$$a_0=[(w_{01}+w_{03})(c_0w_{32}+c_1w_{33})-(w_{31}+w_{33})(w_{02}c_0+w_{03}c_1)]/K$$

$$a_1=[-(2w_{00}+w_{02})(c_0w_{32}+c_1w_{33})+(2w_{30}+w_{32})(w_{02}c_0+w_{03}c_1)]/2K$$

$$a_2=[(w_{01}+w_{03})(-2c_0w_{30}+c_1w_{33})+(w_{31}+w_{33})(2w_{00}c_0+w_{03}c_1)]/2K$$

$$a_3=[(2w_{00}+w_{02})(-c_0w_{32}+c_1w_{31})+(2w_{30}+w_{32})(w_{02}c_0-w_{01}c_1)]/2K$$

Where

$$K=(2w_{30}+w_{32})(w_{01}+w_{03})-(w_{31}+w_{33})(2w_{00}+w_{02})$$

If we put this a_n unknowns in equation (22), we get following equation

$$\rho(t)=a_0+2ta_1+(-2+4t^2)a_2+(-12t+8t^3)a_3$$

This expression is the radius of curvature of the E^3 spherical curve. Therefore, the functions of curvature $\varkappa(t)$ and torsion $\tau(t)$ are obtained by means of this function and its derivatives.

6 Conclusions

In general, the concept of spherical curve on unit velocity curves has been studied [1,17,23,24]. Differently, in this study, the sphericity of arbitrary fast curves in space is examined. Firstly, the second order linear and homogeneous differential equation with variable coefficients, which characterizes arbitrarily fast spherical curves according to Frenet frame in space, is obtained. Then, an approximate solution of the obtained differential equation is found by using the Hermite matrix-collocation method. This value is the radius of curvature of the curve. Using this value, curvature and torsion of the curve can be obtained. Integral characterizations of spherical curves have been obtained [7]. However, such an approximate solution has been presented for the first time in this study. By using the obtained equation and solution, the state of any curve on a sphere can be easily analyzed. Finally, this work can be extended to n-dimensional Euclidean space.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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